The Continuum Hypothesis is True in Practice

Perfect Sets, Descriptive Set Theory, and the Continuum Hypothesis

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Introduction

Most mathematicians are familiar with the story of the Continuum Hypothesis. In its most basic form, the Continuum Hypothesis asserts that if $X \subseteq \mathbb{R}$ is an infinite set, then the cardinality of X is either that of the natural numbers, or that of the entire continuum. That is, $|X| = |\mathbb{N}| = \aleph_0$ or $|X| = |\mathbb{R}| = 2^{\aleph_0}$. This was the first non-trivial problem in cardinal arithmetic, and held the prestigious position as the first on the list of important open problems presented by David Hilbert at the 1900 International Congress of Mathematicians. The "resolution" of the Continuum Hypothesis was a rare combination of fascinating and anti-climactic. Kurt Gödel showed that the Continuum Hypothesis is consistent with ZFC set theory in 1940, and Paul Cohen showed that the negation of the Continuum Hypothesis is consistent with ZFC in 1963. Hence the Continuum Hypothesis became one of the first examples of a "non-contrived" statement independent of ZFC.

While this "standard story" is true, it is incomplete (as so many stories involving Gödel are). After all, there was nearly a century of work between the development of set theory and Cohen's proof of independence! Many early set theorists, including Georg Cantor, believed the Continuum Hypothesis was true! While all attempts to prove it were destined to fail, many fascinating mathematical tools were developed in the interest of a proof. In this essay, we will track one of these historic attempt to resolve the Continuum Hypothesis, sometimes dubbed the "Perfect Set Program". In addition to beautiful math, the Perfect Set Program has a beautiful history. There is value in learning this history! While the Perfect Set Program failed to prove the Continuum Hypothesis, it succeeded in providing new and powerful tools for mathematicians, and in proving a fascinating partial result: **the Continuum Hypothesis is true "in practice"**.

1 Perfect Sets and an Approach to Resolving the Continuum Hypothesis

We start the math with a brief refresher on Perfect sets.

Perfect Sets and their Cardinalities

Definition 1.1: Let $P \subseteq \mathbb{R}$. P is "perfect" if and only if P is closed and every point of P is a limit point.

Let's look at a couple of examples to see the various shapes pefect sets can come in.

Examples 1.2:

1. The empty set is perfect trivially.
2. Every closed interval is perfect.
3. The Cantor set is perfect.

Going forward, while I will do my best to say "non-empty pefect set", we will never be considering empty perfect sets. I may slip up – may God forgive me.

Perfect sets have the following important property:

Proposition 1.3: Let $P \subseteq \mathbb{R}$ be non-empty and perfect. Then $|P| = |\mathbb{R}|$.

Interestingly, most of the proofs of this fact that you can find online and in undergraduate mathematics texts merely show that P is uncountable. Since we're discussing the Continuum Hypothesis though, we don't want to take for granted that every uncountable subset of $\mathbb{R}$ has the same cardinality as all of $\mathbb{R}$! We'll present a true proof that $|P| = |\mathbb{R}|$.

Proof: If P contains a closed interval we're done trivially, so suppose it does not. We make the following important observation: Let $p \in P$, and $a, b \in \mathbb{R} \setminus P$ with $a < p < b$. Then $P \cap [a, b]$ is non-empty and perfect.

The set $P \cap [a, b]$ is trivially nonempty as it contains p, and trivially closed since it's the intersection of two closed sets. Now we check that $P \cap [a, b]$ has no isolated points. Take $x \in P \cap [a, b]$. Since $a, b \notin P$ by hypothesis, we have that $x \in P \cap (a, b)$. For all $\epsilon > 0$ sufficiently small, the open interval $B_\epsilon(x) := (x - \epsilon, x + \epsilon)$ is contained in (a, b). Since $x \in P$ and P is perfect, we know there is a second point $y \in P \cap B_\epsilon(x)$ with $y \neq x$. It follows that x is not an isolated point in $P \cap [a, b]$. Thus $P \cap [a, b]$ has no isolated points and must be perfect.

This observation allows us to embed a Cantor set inside of P. Take two distinct points x_L and x_R in P, taking care to ensure that x_L and x_R are not the least upper bound or greatest lower bound of P (in the event that P is bounded above or below). We may take two disjoint closed intervals $I_L \ni x_L$ and $I_R \ni x_R$ with endpoints in $\mathbb{R} \setminus P$. Set $P_L := P \cap I_L$ and $P_R := P \cap I_R$. These sets P_L and P_R are disjoint, and our observation tells us that both are non-empty and perfect. Now we may repeat this process inside of P_L and P_R to construct four disjoint perfect sets P_{LL}, P_{LR}, P_{RL} and P_{RR}. We may repeat this process inductively. By the Nested Interval Property, every infinite sequence of Ls and Rs corresponds to a non-empty set $P_{LRLLRR\cdots}$. Our construction ensures that no two distinct sequences of Ls and Rs correspond to the same set. Since there are $|2^{\mathbb{N}}| = |\mathbb{R}|$ such sequences, it follows that $|P| = |\mathbb{R}|$. □

Remark 1.4: The Axiom of Choice is not necessary for this proof. All choice functions can be explicitly constructed. The role of choice in the Perfect Set Program will be taken up later.

Remark 1.5: Let 2^ω denote the countable product space $\prod_n \{0, 1\}$, where $\{0, 1\}$ is endowed with the discrete topology. We will refer to this space as the "Cantor Space". Notice that the Cantor Space is homeomorphic to the Cantor set (middle thirds construction), and thus is a compact, totally disconnected, and perfect. In the above proof, what we really did was find a continuous injection $f : 2^\omega \to P$ which is a homeomorphism onto its image. We summarize this with a corollary:

Corollary 1.6: Let P be a non-empty perfect subset of $\mathbb{R}$. Then there is a subset $P' \subseteq P$ which is homeomorphic to the Cantor Space 2^ω. P' is perfect, compact, and totally disconnected.

Perfect Sets and the Continuum Hypothesis

We started this essay with a discussion of the Continuum Hypothesis. What do perfect sets have to do with this? Here's an immediate corollary to Proposition 1.3 to tie things together:

Corollary 1.7: Perfect sets satisfy the Continuum Hypothesis in the sense that if $P \subseteq \mathbb{R}$ is uncountable and perfect, then $|P| = |\mathbb{R}|$.

For the early set theorists, this property of perfect sets hinted at a path toward resolving the Continuum Hypothesis. Perfect sets come in a variety of shapes. They can be continuous like closed intervals, or totally disconnected and nowhere dense like the Cantor set. This flexibility makes it easy to find perfect sets hiding inside of uncountable sets. Wouldn't it be great to come up with an approach to demonstrate the existence of a non-empty perfect subset of every uncountable set of reals? In fact, unless you're a set theorist, I will hazard to guess that every uncountable set of reals you have ever worked with contains a non-empty perfect subset. We summarize the approach with an unreasonable historic goal:

Unreasonable Historic Goal: Show that every uncountable subset of $\mathbb{R}$ contains a non-empty perfect set. This is the so-called "Perfect Set Property" of uncountable subsets of $\mathbb{R}$.

To be clear, this goal is not achievable. The independence of the Continuum Hypothesis from ZFC ensures that we cannot prove all uncountable subsets of $\mathbb{R}$ have the Perfect Set Property, and we will explicitly construct such a set later in this essay.

Ok, so the project is doomed to failure, but we may still find value in tracking this historic attempt at resolving the Continuum Hypothesis. Early set theorists were able to show that broad classes of sets with a given "description" have the Perfect Set Property. For example, every open subset of $\mathbb{R}$ contains a closed interval, which is perfect. This was the beginning of classical "descriptive set theory". While the specific project was futile, you might be surprised to learn just how many sets we're able to prove have the Perfect Set Property. While early set theorists obviously weren't able to prove the Continuum Hypothesis, the success of this "Perfect Set Program" was taken as "empirical evidence" that the Continuum Hypothesis is true. As mentioned before, very few mathematicians ever work with an uncountable subset of $\mathbb{R}$ that doesn't have a non-empty perfect subset. Hence the title of this post: **the Continuum Hypothesis is true in practice.** We revise our unreasonable historic goal to a more modern, reasonable goal:

Reasonable Modern Goal: Show that every uncountable subset of $\mathbb{R}$ that a non-set theorist will ever encounter in the wild has a non-empty perfect subset.

2 Closed Sets and Cantor-Bendixson Analysis

The first big result in the Perfect Set Program was at the hands of Georg Cantor and Ivar Bendixson, who showed that every uncountable closed set contains a non-empty perfect subset. The final proof of this fact was recorded in a letter from Bendixson to Cantor. To get started on understanding this proof, we begin with the notion of the "derived set" from elementary real analysis.

Derived Sets

Definition 2.1: Let $X \subseteq \mathbb{R}$. The derived set of X is $X' := \{a \in \mathbb{R} : a \text{ is a limit point of } X\}$.

Notice that $X \subseteq \mathbb{R}$ is closed if and only if $X' \subseteq X$. Further, X is perfect if and only if $X' = X$. Let's look at some examples:

Examples 2.2:

1. Let $X_0 := \emptyset$. Then $X_0' = \emptyset' = \emptyset$
2. Let $X_1 := \mathbb{N}$. Then $X_1' = \mathbb{N}' = \emptyset$
3. Let $X_2 := \mathbb{N} \bigcup \{n + \frac{1}{2^{m+1}} : n, m \in \mathbb{N}\}$. So X_2 is the naturals, with a sequence of isolated points approaching each natural number n from the right. In this case we, have that $X_2' = \mathbb{N}$. Taking the derived set leaves $\mathbb{N}$ intact, but removes these sequences of isolated points we added. These sequences themselves are removed, but they "protect" the points in $\mathbb{N}$.
4. Let $X_3 := X_2 \bigcup \{x + \frac{1}{3^{k+1}} : x \in X_2 \setminus \mathbb{N} \text{ and } k \in \mathbb{N}\}$. When we moved from $\mathbb{N}$ to X_2, we added in a sequence of points approaching each natural number from the right. Now when we move from X_2 to X_3, we're adding a sequence of points converging to each of those new points in X_2 from the right. These new points "protect" all of the points in X_2, giving us $X_3' = X_2$.
5. Iterating this construction, we may construct a sequence of sets X_n such that X_0 is perfect, and $X_{n+1}' = X_n$ for all n.

Cantor-Bendixson Analysis

The previous examples hint at a question: for a set $X \subseteq \mathbb{R}$, does iteratively taking the derived set always eventually stabilize to a perfect set? If so, after how many times? Will the perfect set be empty? Can there be infinite chains of derived sets like $X \neq X' \neq X'' \neq \cdots$? We will now examine this question for closed X.

Restricting our attention to closed sets makes these questions much more approachable due to the following lemma:

Lemma 2.3: Let $X \subseteq \mathbb{R}$ be closed. Then X' is a closed subset of X.

Proof: Since X is closed, it contains all of its limit points, making $X' \subseteq X$ immediately. Now suppose $\{x_n\}$ is a sequence of points in X' such that $x_n \to x$ for some $x \in X$. x is a limit point of X, so $x \in X'$, making X' closed. $\square$

With this lemma, we may now use the ordinals and transfinite recursion to gain insight. If you need a refresher on ordinals, don't worry. We won't be doing anything too crazy with them, and they won't come up again until the very end. We use ordinals to define the Cantor-Bendixson Derivative and Cantor-Bendixson Rank.

Definition 2.4: Let $X \subseteq \mathbb{R}$ be a closed set. For any ordinal γ, the γ'th Cantor-Bendixson Derivative of X, $\delta^\gamma(X)$, is defined by transfinite recursion as follows:

$$\begin{aligned} \delta^0(X) &:= X \\ \delta^{\alpha+1}(X) &:= \delta^\alpha(X)' \text{ for successor ordinal } \alpha+1 \\ \delta^\beta(X) &:= \bigcap_{\alpha<\beta} \delta^\alpha(X) \text{ for limit ordinal } \beta \end{aligned}$$

Notice that our lemma implies that $\delta^\gamma(X)$ will be closed for every ordinal γ. We next claim that the Cantor-Bendixson derivative always stabilizes eventually.

Proposition 2.5: Let $X \subseteq \mathbb{R}$ be closed. Then there is an ordinal α such that $\delta^\alpha(X) = \delta^{\alpha+1}(X)$.

Proof: For sake of contradiction, suppose the Cantor-Bendixson derivative never stabilizes for X. This implies that for any ordinals $\alpha < \beta$, we have that $X_\alpha \subsetneq X_\beta$. That is, we're throwing out at least one point from X every time we take the derived set. Let γ be an ordinal such that $|\gamma| \gneq |X|$. Then $\delta^\gamma(X) = \emptyset$, as there are more than $|X|$ many ordinals smaller than γ, and we must have thrown out a point every step of the way! Then $\delta^{\gamma+1}(X) = \emptyset = \delta^\gamma(X)$, meaning X stabilizes. This is a contradiction, proving the result. $\square$

Since the Cantor-Bendixson derivative must eventually stabilize at some ordinal, and the ordinals are well-ordered, there must be a smallest ordinal at which we stabilize. This allows us to define the Cantor-Bendixson Rank:

Definition 2.6: Let X be a closed subset of $\mathbb{R}$. The "Cantor-Bendixson Rank" of X is the least ordinal α such that $\delta^\alpha(X) = \delta^{\alpha+1}(X)$. We will denote this by $\text{rk}(X)$.

With rank in hand, we can start moving toward the Cantor-Bendixson Theorem.

Lemma 2.7: Let X be a closed subset of $\mathbb{R}$. Then $\text{rk}(X)$ is a finite or countable ordinal.

Proof: Recall that the topology on $\mathbb{R}$ has a countable basis given by open intervals with rational endpoints. Denote this basis by $\mathcal{B}$. Since X and all its Cantor-Bendixson derivatives are closed, their compliments are open and can be written as a union of the basis intervals. To this end, for every ordinal α, take:

$$\mathcal{B}_\alpha := \{I \in \mathcal{B} : I \subseteq \mathbb{R} \setminus \delta^\alpha(X)\}$$

Notice for ordinals $\alpha \lneq \beta \leq \text{rk}(X)$, the inclusion $\delta^\alpha(X) \supsetneq \delta^\beta(X)$ tells us that $\mathcal{B}_\alpha \subsetneq \mathcal{B}_\beta$. This gives us a chain of proper inclusions:

$$\mathcal{B}_0 \subsetneq \mathcal{B}_1 \subsetneq \cdots \subsetneq \mathcal{B}_{\text{rk}(X)}$$

Since there is a member of this chain for every ordinal $\alpha \leq \text{rk}(X)$, and $\mathcal{B}_{\text{rk}(X)} \subseteq \mathcal{B}$ is countable, it follows that $\text{rk}(X)$ must be a finite or countable ordinal. $\square$

Lemma 2.8: Let A be any subset of $\mathbb{R}$. The set of isolated points of A is finite or countable.

Proof: Let a be an isolated point of A. There is an open interval I with rational endpoints such that $A \cap I = \{a\}$. This gives an injection from the set of isolated points of A into the set of open intervals with rational endpoints. $\square$

We're now ready to state and prove the first big result of the Perfect Set Program: the Cantor-Bendixson Theorem. The proof we present here is (an approximation of) Bendixson's original proof. This proof was written before much of point-set topology had been developed. A more direct proof using the "modern tools" of the early 1900s instead of ordinals will be presented later.

Theorem 2.9 (Cantor-Bendixson Theorem): Let X be a closed subset of $\mathbb{R}$. There is a unique decomposition of X into a disjoint union of a (possibly empty) perfect set and an at most countable set.

Proof: Set $P := \delta^{\text{rk}(X)}(X)$. P is a perfect subset of X. Notice that $X \setminus P$ can be written as the union of the all the isolated points we threw out along the way. That is:

$$X \setminus P = \bigcup_{\alpha \leq \text{rk}(X)} \delta^{\alpha+1}(X) \setminus \delta^{\alpha}(X)$$

Lemmas 2.7 and 2.8 tell us that $\text{rk}(X)$ and each $\delta^{\alpha+1}(X) \setminus \delta^{\alpha}(X)$ are at most countable. Thus $X \setminus P$ is at most a countable union of countable sets, which is countable. So $X = P \cup (X \setminus P)$ is our desired decomposition. The problem of uniqueness will be taken up later. $\square$

Consequences of the Cantor-Bendixson Theorem

We present a few immediately corollaries to the Cantor-Bendixson Theorem. The first two are relevant to the Perfect Set Program and the Continuum Hypothesis.

Corollary 2.10: The closed sets in $\mathbb{R}$ have the "Perfect Set Property", meaning that every uncountable closed $X \subseteq \mathbb{R}$ contains a non-empty perfect subset.

Corollary 2.11: The closed sets of $\mathbb{R}$ satisfy the Continuum Hypothesis. That is, if X is a closed uncountable subset of $\mathbb{R}$, then $|X| = |\mathbb{R}|$.

Proof: Let X be an uncountable closed subset of $\mathbb{R}$. The Cantor-Bendixson Theorem tells us we may decompose X as $X = P \cup C$, where P is perfect, C is at most countable, and $P \cap C = \emptyset$. Since X is uncountable and C is countable, P must be non-empty. By Proposition 1.3, this non-empty perfect set P has the same cardinality as $\mathbb{R}$. Ergo $|X| = |\mathbb{R}|$ $\square$.

The Cantor-Bendixson Theorem has an additional bizarre corollary. It won't be important going forward, but it's interesting.

Corollary 2.12: Assume the negation of the Continuum Hypothesis. Let $A \subseteq \mathbb{R}$ be a Lebesgue-measurable set with an intermediate cardinality $\aleph_0 < |A| < 2^{\aleph_0}$. Then $m(A) = 0$.

Proof: Suppose $m(A) > 0$. Since A is measurable, we may approximate it from the inside be closed sets. Take any closed set $C \subseteq A$ such that $m(C) > 0$. Then C is uncountable, and hence has cardinality of the continuum by Corollary 2.11. This contradicts the fact that $|A| < |\mathbb{R}|$. Therefore $m(A) = 0$. $\square$

Remark 2.13: This result is about Lebesgue measure – NOT Lebesgue outer measure. We need to assume the measurability of the set of intermediate cardinality for Corollary 2.12 to apply. There is no general similar result for outer measure. Whether or not there can be a set $A \subseteq \mathbb{R}$ with

$\aleph_0 < |A| < 2^{\aleph_0}$ and $m_*(A) > 0$ depends on the specific model of ZFC we're using. This is taken up in much more detail in chapter 26 of Thomas Jech's textbook on Set Theory.

3 Unions, Intersections, and Polish Spaces:

Let's take inventory. So far, we've used perfect sets to show that open sets and closed sets in $\mathbb{R}$ satisfy the Continuum Hypothesis. For open sets, this proof was trivial. For closed sets, we had to work a bit harder and introduce the Cantor-Bendixson derivative. We're still very far away from my claim that you've never seen a set without the Perfect Set Property! Let's push further.

Recall that F_σ sets are a class of sets slightly more general than both open and closed sets. A set $X \subseteq \mathbb{R}$ is an F_σ set if and only if X may be written as a countable union of closed sets. It is clear that every open set and closed set is an F_σ set.

Theorem 3.1: F_σ sets in $\mathbb{R}$ have the Perfect Set Property. That is, if $X \subseteq \mathbb{R}$ is an uncountable F_σ set, then there is a non-empty perfect subset $P \subseteq X$.

Proof: Since X is F_σ, we may write $X = \bigcup_n C_n$ where each C_n is closed. Since X is uncountable, at least one of the C_n must be uncountable. Then our previous work with the Cantor-Bendixson Theorem gives us that that there is a non-empty perfect subset $P \subseteq C_n \subseteq X$. $\square$

Corollary 3.2: F_σ sets in $\mathbb{R}$ satisfy the Continuum Hypothesis.

Ok, so we've got the Continuum Hypothesis for countable unions of closed sets. What about countable intersections of open sets? These are the so-called G_δ sets, which generalize open and closed sets in a slightly different way than F_σ sets.

William Young, best known for Young's Inequality, managed to prove this result in 1903. Notice that it was easy to show that both open sets and F_σ sets satisfy the Continuum Hypothesis. Meanwhile, it was pretty tricky to show that closed sets satisfy the Continuum Hypothesis. This trend will continue in that G_δ sets will be tricky as well. Thankfully, we've already done most of the conceptual work. We'll just need to generalize the Cantor-Bendixson Theorem slightly.

Polish Spaces

Definition 3.3: Let X be a topological space. X is a "Polish space"[1] if and only if X is separable and metrizable with a complete metric.

Notice that $\mathbb{R}$ with its standard topology is a Polish space. In fact, if we "forget" the order properties and the distance function of $\mathbb{R}$, but remember the topology, we're left with just a Polish space. Each $\mathbb{R}^n$ with the standard topology is Polish as well. Further, we may still talk about derived sets and perfect sets inside Polish spaces. This hints that Polish spaces behave a lot like $\mathbb{R}$. This is true! As a first example, consider the following lemma:

Lemma 3.4: Let X be a Polish space, and let $P \subseteq X$ be non-empty and perfect. Then $|P| = 2^{\aleph_0} = |\mathbb{R}|$.

The proof of this lemma goes exactly as it did in the case of the reals. We again construct a continuous injection of the Cantor Space 2^ω into P which is homeomorphic onto its image. We just avoid reference to the order properties of $\mathbb{R}$. We also give a Polish-space analogue to Corollary 1.6.

Corollary 3.5: Let P be a non-empty perfect subset of a Polish space X. Then there is a subset $P' \subseteq P$ which is homeomorphic to the Cantor Space 2^ω. P' is perfect, compact, and totally disconnected.

[1]According to Wikipedia, Polish spaces got their name because they were extensively studied by Polish mathematicians, including Tarski. I'm not sure how I feel about this term. Maybe we should make a new term for this? I'm leaving it as "Polish space" for now, but I may change this sometime in the future.

These similarities to $\mathbb{R}$ hint that there might be generalization of the Cantor-Bendixson Theorem. Indeed there is!

Theorem 3.6 (Cantor-Bendixson for Polish Spaces): Let C be a closed subset of a polish space X. There is a unique decomposition of C into a disjoint union of a (possibly empty) perfect set P and an at most countable set S

Recall that our proof of the Cantor-Bendixson Theorem in the last section was the historic proof. Bendixson's argument was from a time before we had the tools of point-set topology. There is a cleaner, more modern way to prove it that generalizes to an arbitrary Polish space. We present that proof here:

Proof: Let X be a Polish space, and $C \subseteq X$ be closed. Now take:

$$P := \{x \in C : \text{for all open } U \ni x, U \cap C \text{ is uncountable}\}$$

Let $S := C \setminus P$. That is, $s \in S$ if and only if there is a neighborhood $U \ni s$ such that $U \cap C$ is countable.

First, we show that P is perfect. If P is empty, we're done. Suppose $P \neq \emptyset$. It is obvious from the definition that every point of P is a limit point of P. Meanwhile, suppose that $x \in X$ is a limit point of P. Since C is closed, we know that $x \in C$. Let $U \ni x$ be open. Since x is a limit point of P, there is a point $p \in P \cap U$. Hence, there are uncountably many points of P in U. Since U was an arbitrary open set containing x, we have that $x \in P$ and P closed.

Next we show that S is countable. X is metrizable and separable, so X is second countable as well. Thus we have a countable basis for the topology on X. Around each $s \in S$, we may take a not-necessarily distinct basis neighborhood U_s such that $U_s \cap C$ is countable. Then $\bigcup_{s \in S}(U_s \cap C)$ is a countable union of countable sets. We're working in ZFC, so this set is countable. Further, $S \subseteq \bigcup_{s \in S}(U_s \cap C)$, making S countable as well.

This proves that $C = P \cup S$ is decomposition of closed C into the disjoint union of a perfect set P and a closed set S.

To prove this decomposition is unique (missing in the last proof), suppose that $C = P' \cup S'$ is another such decomposition. Let $x \in P'$. Let $U \ni x$ be an open set, and take another open $V \ni x$ such that $\overline{V} \subseteq U$. I claim that $\overline{V \cap P'}$ is perfect. The set is clearly closed, so we merely need to show that every point is a limit point. Let $y \in \overline{V \cap P'}$. Then for any neighborhood $N \ni y$, we have that $N \cap V \cap P'$ is non-empty. Take $z \in N \cap V \cap P'$. Then z is a limit point of P'. Since $N \cap V$ is open, it follows that there is a second point $z' \in N \cap V \cap P'$ with $z \neq z'$. y is a limit point of $\overline{V \cap P'}$. Hence $\overline{V \cap P'}$ is perfect, and has cardinality $2^{\aleph_0}$. So U has cardinality $2^{\aleph_0}$ as well. Since U was an arbitrary neighborhood of x, it follows that $x \in P$. This implies that $P' \subseteq P$.

On the other hand, suppose that $x \in S'$. since P' is closed, we may find a neighborhood $U \ni y$ such that $U \cap P' = \emptyset$. Hence $U \cap C \subseteq S'$, so $U \cap C$ is countable. Hence $s' \in S$, making $S' \subseteq S$. It follows that $P = P'$ and $S = S'$, making the decomposition unique. □

Remark 3.7: The uniqueness result here fills in the missing part of the proof of the Cantor-Bendixson Theorem in $\mathbb{R}$.

Corollary 3.8: Every uncountable Polish space has the cardinality of the continuum.

Proof: Let X be an uncountable Polish space. Then X is a closed subset of itself, so we may write $X = P \cup C$, where P is perfect and C is at most countable. Hence $|X| = 2^{\aleph_0} = |\mathbb{R}|$. □

Before we get back to the story, we introduce a utility lemma. We give it this name because it will finish several proofs for us along the way.

Lemma 3.9 (Utility lemma): Let 2^ω be the Cantor Space, X be a Polish space, and $f : 2^\omega \to X$ be a continuous injection. Then f is a homeomorphism onto its image.

Proof: Let $P := f(2^\omega)$ be the image of f equipped with the subspace topology from X. We wish to show that $f : 2^\omega \to P$ is a homeomorphism. All we must show is that $f^{-1} : P \to 2^\omega$ is continuous. It will suffice to show that f maps closed sets to closed sets. Take a closed $C \subseteq 2^\omega$. Since C is closed and 2^ω is compact, it follows that C is compact. Since f is continuous, we get that $f(C) \subseteq P$ is compact. Notice that P is Hausdorff, since P is equipped with the subspace topology from the Polish topology on X. Hence $f(C) \subseteq P$ is closed. Therefore f is a homeomorphism onto its image. □

G_δ Sets In Polish Spaces Are Polish Spaces

Recall we were working toward showing that G_δ sets in $\mathbb{R}$ have the Perfect Set Property. The last missing step is Alexandrov's Theorem:

Theorem 3.10 (Alexandrov's Theorem): Let X be a Polish space. If A is a G_δ subset of X, then A equipped with the subspace topology is a Polish space.

Remark 3.11: The converse is also true.

The proof of Alexandrov's Theorem is pretty boring metric space topology work. A full proof is included here for good measure, but I would personally recommend skipping ahead to Theorem 3.13. We start the proof with a lemma:

Lemma 3.12: Let X be a Polish space. Let $U \subseteq X$ be open. Then U equipped with the subspace topology is Polish.

Proof: Since X is separable, it immediately follows that U is separable in the subspace topology. We only need to show that U may be metrized by a complete metric. Let d be a complete metric that metrizes the topology on X. An immediate issue is that Cauchy sequences in U with respect to d need not converge to a point in U. Hence we will need to build a new metric.

First, notice that we may, without loss of generality, assume that $d(x, y) < 1$ for all $x, y \in X$. If not, replace d with the new metric $\frac{d(x,y)}{1+d(x,y)}$. This new metric is bounded above by 1 and is compatible with the same topology on X.

Consider the map $d' : U \times U \to \mathbb{R}_{\geq 0}$ given by:

$$d'(x, y) = d(x, y) + \left| \frac{1}{d(x, X \setminus U)} - \frac{1}{d(y, X \setminus U)} \right|$$

Since $X \setminus U$ is closed, this map is well defined for $x, y \in U$. I claim that d' is a metric on U. Most properties are immediately apparent, but we will explicitly check the triangle inequality. Let $x, y, z \in U$. Then:

$$\begin{aligned} d'(x, z) &= d(x, z) + \left| \frac{1}{d(x, X \setminus U)} - \frac{1}{d(z, X \setminus U)} \right| \\ &\leq d(x, y) + d(y, z) + \left| \frac{1}{d(x, X \setminus U)} - \frac{1}{d(y, X \setminus U)} \right| + \left| \frac{1}{d(y, X \setminus U)} - \frac{1}{d(y, X \setminus U)} \right| \\ &= d'(x, y) + d'(y, z) \end{aligned}$$

To complete the proof, we must verify that this metric d' is complete and compatible with the subspace topology on U.

d' is a complete metric: Suppose that $\{x_n\}_n$ is a Cauchy sequence in U with respect to d'. Since $d'(x_n, x_m) \geq d(x_n, x_m)$, the sequence is Cauchy with respect to the complete metric d as well. Hence there is $x \in X$ such that $x_n \to x$. We must show that $x \in U$. Notice that since

$d'(x_n, x_m) \to 0$ and $d(x_n, x_m) \to 0$ as $m, n \to 0$, it follows from the definition of d' that:

$$\left| \frac{1}{d(x_n, X \setminus U)} - \frac{1}{d(x_m, X \setminus U)} \right| \to 0$$

as $m, n \to \infty$. Hence the sequence $\frac{1}{d(x_n, X \setminus U)}$ is Cauchy in $\mathbb{R}$, and thus converges to some $r \in \mathbb{R}$. Since $d < 1$, we know that $r > 0$. By the reverse triangle inequality, we may bound:

$$d(x, X \setminus U) \geq |d(x, x_n) - d(x_n, X \setminus U)|$$

Taking the limit as $n \to \infty$, we get:

$$d(x, X \setminus U) \geq \frac{1}{r} > 0$$

Hence $x \in U$, proving that d' is complete.

d' is compatible with the subspace topology on U: It suffice to check that if $x \in U$ and $\{x_n\}_n$ is a sequence of points in U, then $x_n \to x$ with respect to d' if and only if $x_n \to x$ with respect to d. First suppose that $x_n \xrightarrow{d'} x$. Since $d(x_n, x) \leq d'(x_n, x)$, it follows that $x_n \xrightarrow{d} x$. Conversely, suppose $x_n \xrightarrow{d} x$. By continuity, we know that $d(x_n, X \setminus U) \xrightarrow{d} d(x, X \setminus U)$. Since x and each x_n is in U, we know that every term in this sequence and the limit are non-zero. Hence we may invert to get:

$$\frac{1}{d(x_n, X \setminus U)} \longrightarrow \frac{1}{d(x, X \setminus U)}$$

It follows that:

$$d'(x_n, x) = d(x_n, x) + \left| \frac{1}{d(x_n, X \setminus U)} - \frac{1}{d(x, X \setminus U)} \right| \longrightarrow 0$$

Hence $x_n \xrightarrow{d'} x$. This completes the proof. □

With this lemma, we may prove Alexandrov's Theorem.

Proof of Alexandrov's Theorem: Let A be a G_δ set contained in a Polish space X. As before, since A is a subspace of a separable space, A is separable. Since A is G_δ, we may write $A = \bigcap_n U_n$, where $U_n \subseteq X$ is open and $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. By the lemma, each U_n is a Polish space, and hence is metrizable by a complete metric d_n. We may again, without loss of generality, assume that each d_n is bounded above by 1. We define a new metric $d : X \times X \to \mathbb{R}_{\geq 0}$ by:

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x, y)$$

Since each d_n is bounded above by 1, this sum converges. It is also immediately clear that d is a metric. First, I claim that d is a complete metric. Let $\{x_k\}_k$ be a Cauchy sequence in A with respect to d. It immediately follows that $\{x_k\}_k$ is a Cauchy sequence with respect to d_n as well for all n. So x_k has a limit x in every U_n. The limit must be the same for each n since each d_n is compatible with the the subspace topology on U_n. Hence $x \in A$ as well, so d is a complete metric. Finally, let $\{x_k\}_k$ be a sequence in A, and let $a \in A$. It is clear that $x_k \xrightarrow{d} a$ if and only if $x_k \xrightarrow{d_n} a$ for all n. So d is compatible with the subspace topology on A. □

Cantor-Bendixson Analysis on G_δ Sets

With Alexandrov's theorem, we may extend Cantor-Bendixson analysis to G_δ sets.

Theorem 3.13: G_δ sets in a Polish space X have the Perfect Set Property.

Proof: Let G be an uncountable G_δ set in X. Since X is a Polish space the Alexandrov Theorem tells us that G equipped with the subspace topology is a Polish space. Since G is an uncountable closed subset of itself as a Polish space, the Cantor-Bendixson Theorem gives us that there is a perfect set $P \subseteq G$ with $|P| = 2^{\aleph_0}$.

Notice this set P is perfect with respect to the subspace topology on G. This does not mean that P is perfect with respect to the topology on G. For example, if we look at $(0, 1)$ as a G_δ set in $\mathbb{R}$, our perfect P would be all of $(0, 1)$. We still have more work to do!

Since P is perfect in the Polish topology on G, there is a continuous embedding of the Cantor Space into P that is homeomorphic onto its image. Take $f : 2^\omega \to P$ to be this embedding. Since the Polish topology on G is finer than the Polish topology on X, f is a continuous injection of 2^ω into X. By our Utility lemma (3.9), f is homeomorphic onto its image. Call this image $P' := f(2^\omega)$. It follows that P' is a perfect subset of X with respect to the original topology on X. $\square$

Corollary 3.14: G_δ sets in $\mathbb{R}$ have the Perfect Set Property, and thus satisfy the Continuum Hypothesis.

4 Borel Sets

Now we're up to F_σ and G_δ sets satisfying the Continuum Hypothesis. What about countable intersections of F_σ sets and countable unions of G_δ sets? What about countable unions and intersections of these sets? If we carry these constructions out ad infinitum, we will arrive at the "Borel σ-Algebra". This is the smallest collection of sets containing the open sets that is closed under the operations of countable union, countable intersection, and complement. Do Borel sets have the Perfect Set Property?

What makes this problem difficult, roughly, is that the Perfect Set Property doesn't play nicely with complements. While one can make statements about countable unions and intersections of sets with perfect subsets, it is very difficult to make statements about whether the complement of a set containing a non-empty perfect set also contains a non-empty perfect set. This makes it difficult to induct through the Borel hierarchy and arrive at the result for all Borel sets.

Indeed the Borel sets do have the Perfect Set Property though! This result was proved by both Alexandrov and Hausdorff in 1916, taking advantage of a different description of the Borel sets. Interestingly, the original proof by Alexandrov is extremely difficult to track down. I could only find it in French on a shady website in a file format specifically for Kindle e-readers. In the interest of preserving history, I risked the malware and have translated the entire paper to English. I am not sure if I would be in violation of copyright law to post the translation online, but feel free to reach out to me if you would like a copy. Sadly though, even in English I cannot really follow the argument of the paper! Bear in mind that the Russian Alexandrov's writing in French, has been translated to English by a person who took exactly two years of French classes in high school. It is very likely that something got lost in translation along the way.

While the historic proof may be interesting, we will provide a different proof here for Borel sets in arbitrary Polish spaces. As in the last section on G_δ-sets, I would recommend simply reading the statements of theorems and propositions, but skipping over the proofs unless you are very interested.

Refining Topologies and Borel Sets

Our work begins with an exploration of the various Polish topologies we may put on a space. It turns out, we may refine a given Polish topology in a way that preserves the Borel sets, but gives a specific set a nicer description with regard to the Topology. That is, we may push a set down the Borel hierarchy. We start with a lemma:

Lemma 4.1: Let τ be a Polish topology on a set X. Let $C \subseteq X$ be closed with respect to τ. Let τ' be the topology generated by $\tau \cup \{C\}$. Then (X, τ') is also Polish, C is clopen with respect to τ', and τ' has the same Borel sets as τ.

Proof: First, it is immediately clear that C is clopen in τ'.

Next, Let τ_C and $\tau_{X \setminus C}$ denote the subspace topologies of τ on C and $X \setminus C$ respectively. Since C is closed with respect to τ, $\tau_{X \setminus C}$ is exactly the open sets of τ contained in $X \setminus C$. Notice that $U \in \tau'$ if and only if both $U \cap (X \setminus C) \in \tau_{X \setminus C}$ and $U \cap C$ is open in τ_C. This characterization makes it clear that (X, τ') is separable because τ is.

Since C and $X \setminus C$ are closed and open respectively in the polish space (X, τ), we know that $(X \setminus C, \tau_{X \setminus C})$ and (C, τ_C) are both Polish spaces. Hence, take d_C and $d_{X \setminus C}$ to be complete metrics compatible with the topologies τ_C and $\tau_{X \setminus C}$ respectively. Without loss of generality, take d_C and $d_{X \setminus C}$ to be bounded above by 1. Take d to be a new metric on X by:

$$d(x,y) = \begin{cases} d_C(x,y) & \text{if } x, y \in C \\ d_{X \setminus C}(x,y) & \text{if } x, y \in X \setminus C \\ 2 & \text{else} \end{cases}$$

Now we check that d is a complete metric on X. In particular, if $\{x_n\}_n$ is a Cauchy sequence in X with respect to d, then it follows that $\{x_n\}_n$ is eventually always inside C or $X \setminus C$. If the sequence is eventually inside C, then $d(x_n, x_m) = d_C(x_n, x_m)$ for all n, m sufficiently large. Since d_C is a complete metric on C, we get that x_n converges. A similar argument holds in $X \setminus C$. Hence d is a complete metric.

Now we show that the metric d is compatible with the topology τ'. First let $\epsilon \in (0,1)$, $x \in X$, and $B_\epsilon(x)$ be the open ϵ-ball centered at x with respect to d. $B_\epsilon(x)$ must be fully contained in C or $X \setminus C$. Suppose it is inside C. Then $d = d_C$, and hence is compatible with the subspace topology on C. We may take a U open in τ such that $U \cap C \subseteq B_\epsilon(x)$. Notice $U \cap C$ is open in τ'. A similar argument holds when $x \in X \setminus C$. We have just proved that τ' is no more course than metric induced by d. Conversely Let $U \in \tau'$. Take $x \in U$. Suppose $x \in C$. Then $U \cap C$ is open in τ_C. Since d_C is compatible with the topology τ_C, we may take an $\epsilon > 0$ such that the ϵ-ball with respect to d_C is contained in $U \cap C$. But this is also the ϵ-ball with respect to d. A similar argument again holds if $x \in X \setminus C$. Hence d is compatible with the topology τ' on X.

At this point we have shown that (X, τ') is a Polish space in which C is clopen. We still must show that τ' has the same Borel sets as τ. It suffice to show that every open set in τ is a Borel set in τ' and vice-versa. Since τ' is a finer topology than τ, every open set in τ is open (and hence Borel) in τ'. Take U open in τ'. Then write $U = (U \cap (X \setminus C)) \cup (U \cap C)$. Since U is open in τ', we know that $U \cap (X \setminus C)$ is open in $\tau_{X \setminus C}$, and $U \cap C$ is open in τ_C. Hence we may take open sets V_1 and V_2 in τ such that $U \cap (X \setminus C) = V_1$ and $U \cap C = V_2 \cap C$. So $U = V_1 \cup (V_2 \cap C)$, which is clearly Borel in τ. Hence τ and τ' have the same Borel sets. □

We may extend this result from closed sets to Borel sets.

Theorem 4.2: Let τ be a Polish topology on a set X, and let $B \subseteq X$ be Borel with respect to τ. There is a finer topology τ_B on X such that τ and τ_B have the same Borel sets, and B is clopen in τ_B.

Proof: Sadly, we won't just be able to add B as an open set this time. We will have to be more subtle. Take $\mathcal{A}$ to be the collection of all sets with the following property: A is in $\mathcal{A}$ if and only if there is a finer topology $\tau_A \supseteq \tau$ such that τ_A has the same Borel sets as τ and A is clopen with respect to τ_A.

First, notice that every $A \in \mathcal{A}$ must be a Borel set with respect to τ. In particular, A is open in τ_A, and hence Borel with respect to τ_A. Since τ_A has the same Borel sets as τ, A must be Borel with respect to τ.

Now we will show that $\mathcal{A}$ contains all the Borel sets. To prove this, it will suffice to show that $\mathcal{A}$ is a σ-algebra containing the open sets of τ.

To see that $\mathcal{A}$ contains the open sets, let U be open in τ. Then lemma 4.1 ensures that there is a finer topology τ' with the same Borel sets such that the closed set $X \setminus U$ is clopen in τ'. So U has an open complement in τ', making U closed. Hence $U \in \mathcal{A}$.

To show that $\mathcal{A}$ is a σ-algebra, it will suffice to check that it is closed under complements and countable unions.

Complements: Let $A \in \mathcal{A}$. Then there is a finer topology τ_A with the same Borel sets as τ in which A is clopen. Then $X \setminus A$ is clopen in τ_A as well, making $X \setminus A$ a member of $\mathcal{A}$.

Countable Unions: Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable family of sets in $\mathcal{A}$. For each n, we may take a τ_n such that A_n is clopen in τ_n, τ_n is finer than τ, and has the same Borel sets. Take τ_∞ to be the topology generated by the countable union $\bigcup_n \tau_n$. We check that this is a Polish topology on X with the same Borel sets as τ.

To see that τ_∞ is Polish, we observe that the countable diagonal map $\Delta : (X, \tau_\infty) \to \prod_{n\in N}(X, \tau_n)$ sending $x \mapsto (x, x, \cdots)$ is a homeomorphism of (X, τ_∞) onto its image. Since a countable products preserve seperability and complete-metrizability, this product space $\prod_n(X, \tau_n)$ is Polish. Consider a point $z = (z_1, z_2, \cdots)$ in the complement of the image $\Delta(X)$. There must be $i < j$ such that $z_i \neq z_j$. Since Polish spaces are Hausdorff, we may take disjoint open sets U, V in τ such that $z_i \in U$ and $z_j \in V$. Since each τ_n is a refinement of τ, it follows that U is open in τ_i and V is open in τ_j. Hence we may take an open set in $\prod_n(X, \tau_n)$ by:

$$W := X \times \cdots \times X \times U \times X \times \cdots \times X \times V \times X \times \cdots$$

where the U and V are in the ith and jth places respectively. Since U and V are disjoint, it is clear that W is completely contained in the complement of $\Delta(X)$. Hence $\Delta(X)$ is closed. Since closed subsets of Polish spaces are themselves Polish in the subspace topology, it follows that $\Delta(X)$ is Polish. Hence, τ_∞ is a Polish topology on X.

Now we check that τ_∞ has the same Borel sets as τ. Since τ_∞ is a finer topology than τ, we know that the Borel sets of τ are contained in the Borel sets of τ_∞. On the other hand, it will suffice to check that every open set in τ_∞ is Borel in τ. Suppose that U is open in τ_∞. For each n, we may take a countable basis $\{V_j^n\}_j$ for the topology τ_n. Each of these V_j^n are Borel with respect to τ, and $\{V_j^n\}_{n,j}$ forms a sub-basis for the topology τ_∞. Hence, we may write our open $U \in \tau_\infty$ as a countable union of finite intersections of these $\{V_j^n\}_{n,j}$, making U is Borel in τ. Thus τ_∞ has the same Borel sets as τ.

At this point, we have that τ_∞ is a Polish topology on X with the same Borel sets as τ. Since $\bigcup_n A_n$ is open in τ_∞, its complement is closed. Applying our lemma to the compliment gives us a finer Polish topology $\tau' \supseteq \tau_\infty$ in which $\bigcup_n A_n$ is clopen, with the same Borel sets as τ_∞, and thus the same Borel sets as τ. Therefore $\mathcal{A}$ is closed under countable unions.

Thus we have shown that $\mathcal{A}$ is exactly the Borel sets of τ, completing the proof. □

Cantor-Bendixson Analysis on Borel Sets

The finer topology given by Theorem 4.2 allows us to apply the Cantor-Bendixson Theorem to Borel sets in Polish spaces. This approach will allow us to continue the Perfect Set Program. We present the following Corollary:

Corollary 4.3: Borel sets in Polish spaces have the Perfect Set Property.

Proof: Let B be an uncountable Borel set in a Polish space (X, τ). By the theorem, there is a finer topology τ' such that (X, τ') is Polish, and B is clopen with respect to τ'. There is an embedding $f : 2^\omega \to B$ homeomorphic onto its image with respect to the subspace topology of τ' on B. Then f is a continuous injection of 2^ω into (X, τ). Our utility lemma (3.9) again gives us that f is a homeomorphic embedding of 2^ω into (X, τ). Hence B contians a non-empty perfect subset. □

Corollary 4.4: Borel sets in $\mathbb{R}$ have the Perfect Set Property, and thus satisfy the Continuum Hypothesis.

5 The Analytic Sets

While every mathematicians comes across the Borel sets at some point, relatively few come across the analytic sets. These sets hold a fun place at the center of one of the biggest "incorrect published proofs" by a prominent mathematician. We start with a definition:

Definition 5.1: Let $A \subseteq \mathbb{R}$. A is an "analytic set" if and only if there is a Borel set $B \subseteq \mathbb{R} \times \mathbb{R}$ such that $A = \pi B$, where π is the first coordinate projection map.

In 1905, Henri Lebesgue published a paper (https://eudml.org/doc/234955) with the following claim (bottom of page 191): If $B \subseteq \mathbb{R}^2$ is Borel in the plane, then its image under the first coordinate projection is Borel as well. This was, in effect, asserting that the analytic sets in $\mathbb{R}$ are the same as the Borel sets. It's clear that every Borel set $B \subseteq \mathbb{R}$ is analytic (simply take the projection of $B \times \mathbb{R}$), but it's not obvious that every analytic set is Borel. On its face, this claim sounds possible though. After all, the projection of an open set is open, and it is clear that projection maps commute with countable unions. It seems reasonable that everything should work out on the σ-algebra generated by the open sets as well. We may recognize a problem in that projection does not commute with countable intersections, but this doesn't necessarily mean that the projection of a countable intersection of Borel sets in $\mathbb{R}^2$ isn't Borel in $\mathbb{R}$. Thus, the existence of an analytic-but-not-Borel set isn't obvious either.

This unjustified claim in Lebesgue's paper was discovered and fleshed out by the Russian mathematician Mikhail Suslin, who was able to demonstrate that the Borel sets and analytic sets are not the same. Finding a set which is analytic but not Borel is tough. The following example is due to Luzin, Suslin's advisor at Moscow State University. Let $x \in \mathbb{R}$ be irrational. Then x has a unique continued fraction expansion:

$$x = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}$$

Where each $a_n \in \mathbb{Z}$, and $a_n > 0$ for every $n > 0$. Identify each real number with its corresponding sequence of integers. Let A be the set of all irrational numbers such that the corresponding sequence $(a_0, a_1, a_2, \cdots)$ has an infinite subsequence $(a_{n_1}, a_{n_2}, a_{n_3}, \cdots)$ satisfying $a_{n_j} \mid a_{n_{j+1}}$ for all j. A is analytic, but not Borel.

We start our exposition by extending our definitions to arbitrary Polish spaces.

Definition 5.2: Let X be a Polish space, and $A \subseteq X$. We say that A is "analytic" if and only if there is a Polish space Y and a Borel set $B \subseteq X \times Y$ such that $A = \pi B$, where π is the first coordinate projection.

Remark 5.3: Definitions 5.2 for analytic sets in Polish spaces looks different than than definition 5.1 for an analytic set in $\mathbb{R}$. In particular, it is not clear that we may assume the arbitrary Polish space Y in definition 5.2 can be taken to be $\mathbb{R}$. Indeed though, we may always pick Y to the another copy of the Polish space X. Both of these definitions of "analytic" are equivalent, and are given in different books. Proving these are equivalent is quite challenging, requiring use of the Borel Isomorphism Theorem–a fundamental theorem of measure theory lives at the heart of probability and stochastic processes. A lengthy proof and discussion of why these definitions are equivalent is included in the appendix. Perhaps surprisingly, we will need both of these definitions and other equivalent characterizations to proceed.

Equivalent Characterizations of Analytic Sets

We start with two new and useful characterizations of analytic sets:

Theorem 5.4: Let A be a subset of a Polish space X. The following are equivalent:

1. A is analytic.

2. A is the continuous image of a Borel set in a Polish space Y.

3. A is empty or the continuous image of the "Baire Space" $\mathcal{N}$, whose points are infinite sequences in $\mathbb{N}$ endowed with the product topology on countably many copies of $\mathbb{N}$ as a discrete space.

Proof:
$1 \implies 2$*:* Let $A \subseteq X$ be analytic. Then there is a Polish space Y and a Borel set $B \subseteq X \times Y$ such that $A = \pi B$. Since $X \times Y$ is a Polish space and $\pi : X \times Y \to X$ is continuous, we see that A is the continuous image of a Borel set in a Polish space.

$2 \implies 3$*:* Suppose $A \neq \emptyset$. Let $A = f(B)$ where $f : Y \to X$ is continuous, Y is Polish, and $B \subseteq Y$ is Borel. By composing maps, it will suffice to show that there is a continuous surjection $g : \mathcal{N} \to B$. Recall that by Theorem 4.2, we may refining the topology on Y in such a way that B becomes clopen in Y. Thus, we may view B itself as a Polish subspace of Y by Alexandrov's Theorem (3.10). Since this topology on Y is finer than the original topology, f is still continuous. Hence it will suffice to show that if Z is a Polish space, we may find a continuous surjection $g : \mathcal{N} \to Z$.

To show this result, let *Seq* denote the set of all finite sequences of natural numbers, and let $\frown$ denote concatination of sequences. Let Z be a Polish space with complete metric d such that $d \leq 1$. Let $D \subseteq Z$ be a countable dense subset. We inductively define a collection of open balls $\{C_s : s \in S\}$ as follows. Let $C_\emptyset := Z$. Now suppose we know C_s for some $s \in Seq$. Let $\{x_k\}_{k \in \mathbb{N}}$ is an enumeration of $C_s \cap D$ Given a ball C_s, take $C_{s \frown k}$ to be the open ball centered at x_k with radius $\min\{\text{length}(s \frown k)^{-1}, \sup\{r > 0 : B(x_k, r) \subseteq C_s\}\}$, where $B(x_k, r)$ denotes the open ball centered at x_k with radius r.

We observe that $\text{diam}(C_s) \leq \frac{1}{\text{length}(s)}$, that $C_s = \bigcup_k C_{s \frown k}$, and that if s is a prefix of t in S (denoted $s \leq t$), then $\text{center}(C_t) \in C_s$.

Now we define our map $g : \mathcal{N} \to Z$. For $a = (a_1, a_2, a_3, \cdots)$ in $\mathcal{N}$, we take the value:

$$f(a) = \bigcap_{s \leq a} \overline{C_s}$$

This intersection is taken over all prefixes of $a = (a_1, a_2, a_3, \cdots)$. By our construction, it is clear this map is well defined, continuous, and surjective. This proves the result.

$3 \implies 1$*:* If $A = \emptyset$, we're done trivially. Suppose $A = f(\mathcal{N})$ where f is continuous. Consider the graph of f (with the coordinates reversed), given by:

$$\Gamma(f) := \{(x, y) \in X \times \mathcal{N} \; : \; x = f(y)\}$$

It is clear that $\pi\Gamma(f) = A$ and $\Gamma(f)$ is closed, so it will suffice to show that $\mathcal{N}$ is a Polish space. To see this is true, consider the map $\phi : \mathcal{N} \to [0, 1] \setminus \mathbb{Q}$ given by:

$$\phi(a_1, a_2, a_3, \cdots) := \frac{1}{(a_1 + 1) + \frac{1}{(a_2+1)+\frac{1}{(a_3+1)+\cdots}}}$$

Since such continued fraction representations are unique, we see that ϕ is a bijection. When viewing $[0, 1] \setminus \mathbb{Q}$ as a subspace of $[0, 1]$, it is easy to see that ϕ is a homeomorphism. Finally, recognize that if we take $\{q_n\}_n$ to be an enumeration of the rationals in the interval $[0, 1]$, we may write $[0, 1] \setminus \mathbb{Q} = \bigcap_n [0, 1] \setminus \{q_n\}$. This shows that $[0, 1] \setminus \mathbb{Q}$ is a G_δ subset of the Polish space $[0, 1]$. Hence $[0, 1] \setminus \mathbb{Q}$ is a Polish space with the subspace topology, showing $\mathcal{N}$ is a Polish space. □

Remark 5.5: This characterization reveals that the Borel sets are not closed under continuous images.

Topology to Topiary: the Basics of Trees

Our next goal will be to prove that analytic sets have the Perfect Set Property. The language of trees will be useful for this argument.

Definition 5.6: Let $(T, <)$ be a partially ordered set. $(T, <)$ is a "tree" if and only if the set of predecessors $\{y \in T : y < x\}$ is well ordered. Given a set A, a "tree on A" is a collection of finite-length tuples $T \subseteq A^{<\omega}$ that is closed under initial segments.

Notation 5.7: We introduce a little necessary notation. If $t \in A^n$ and $s \in A^m$, then we write $t \preceq s$ if t is a prefix (initial segment) of s. Hence, a tree on A is just a collection $T \subseteq A^{<\omega}$ with the property that if $s \in T$ and $t \preceq s$, then $t \in T$. Further, if T is a tree on A, we use $[T]$ to denote the collection of infinite paths through T. That is, $[T] = \{p \in A^\omega : t \in T \text{ for all } t \preceq p\}$.

We make some elementary observations about trees. We will be primarily concerned about trees on $\mathbb{N}$:

Proposition 5.8:

1. A tree T on $\mathbb{N}$ is a subseteq of Seq, and $[T]$ is a subset of the Baire Space $\mathcal{N}$.
2. If T_1 and T_2 are trees on $\mathbb{N}$, the intersection $T_1 \cap T_2$ is a tree on $\mathbb{N}$.
3. If T is a tree on $\mathbb{N}$ and $s \in Seq$, then we may build a new tree on $\mathbb{N}$ by taking only points of T which are on branches that pass through s. That is, we get a tree $T_s := \{t \in T : t \preceq s \text{ or } s \preceq t\}$
4. F is a closed set in the Baire Space $\mathcal{N}$ if and only if there is a tree T on $\mathbb{N}$ such that $F = [T]$.
5. Let T be a tree on $\mathbb{N}$. Call $t \in T$ a "leaf" iff there is no $s \in T$ such that $t \prec s$. T has no leaves if and only if for every $t \in T$ there is an infinite path $p \in [T]$ such that $t \prec p$.

Proof:

1. If T is a tree on $\mathbb{N}$, then $T \subseteq \mathbb{N}^{<\omega}$. Recall that Seq is just another name for finite sequences of natural numbers. So $T \subseteq Seq$. Further if p is an infinite path through T, then p is an infinite sequence of natural numbers in $\mathbb{N}^\omega = \mathcal{N}$.
2. If T_1 and T_2 are trees on $\mathbb{N}$, then $T_1, T_2 \subseteq Seq$. Hence $T_1 \cap T_2 \subseteq Seq$. Further suppose that $s \in T_1 \cap T_2$, and $t \preceq s$. Then since T_1 and T_2 are trees, $t \in T_1$ and $t \in T_2$. It follows that $t \in T_1 \cap T_2$ making $T_1 \cap T_2$ a tree.
3. We may identify $T_s = T \cap \{t \in Seq : t \preceq s \text{ or } s \preceq t\}$. The latter set in this intersection is clearly a tree, so the result follows from part 2.
4. Suppose $F \subseteq \mathcal{N}$ is closed. Set $T_F := \{s \in Seq : s \preceq f \text{ for some } f \in F\}$. It is clear that T_F is a tree and $F \subseteq [T_F]$. Suppose that $p \in [T_F]$. Let $p_n \in Seq$ denote the initial segment of p given by the first n points in the sequence. For each p_n, we may take an $f_n \in F$ such that $p_n \prec f_n$. For $m, n \geq k$, we know that the sequences f_m and f_n must agree on at least the first k places. Hence we see that f_n is a Cauchy sequence. Since $\mathcal{N}$ is complete, it follows that $f_n \to f$ for some $f \in \mathcal{N}$. Since F is closed, it follows that $f \in F$. Hence $[T_F] = F$.

   Conversely suppose that $F = [T]$ for some tree T on $\mathcal{N}$. If $[T] = \emptyset$ we're done trivially. So suppose $[T]$ is non-empty. Let $f \in \mathcal{N}$ be a point such that $f \notin [T]$. Let $f_n \in Seq$ denote the initial segment of f given by taking the first n points in the sequence. Since $f \notin [T]$, there is an n such that $f_n \notin T$. Let $U := \{g \in \mathcal{N} : f_n \prec g\}$. It is clear that U is open in $\mathcal{N}$ and $f \in U$. Further, since T is a tree it is easily seen that $[T] \cap U = \emptyset$. Thus the complement of $[T]$ is open, making $[T]$ closed in $\mathcal{N}$.
5. Let T be a tree on $\mathbb{N}$. Suppose that T has no leaves. Then for $t \in T$, set $p_0 = t$ and for every n, take $p_n \in T$ such that $p_n \prec p_{n+1}$. Since T has no leaves, such a sequence exists. Further it is clear that there is a $p \in \mathcal{N}$ such that $p_n \prec p$ for each n. Hence there is an infinite path p such that $t \prec p$.

   Conversely, suppose that T has a leaf. Let ℓ be a leaf of T. Then there is no $t \in T$ such that $\ell \preceq t$. Hence there can be no $p \in [T]$ such that $t \prec p$. $\square$

Cantor-Bendixson Analysis on Analtyic Sets

Theorem 5.9: The analytic sets in Polish spaces satisfy the Perfect Set Property.

To prove this theorem, we copy our original approach for closed sets. In particular, we must define the Cantor-Bendixson derivative and rank for analytic sets.

Construction 5.10: Let A be an analytic set in a Polish space X. By Theorem 5.4, we know that A is the continuous image of the Baire Space $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ equipped with the product topology, viewing $\mathbb{N}$ as a discrete topological space. Let f be a continuous function such that $f(\mathcal{N}) = A$.

Let T be a tree on $\mathbb{N}$. T is a subset of Seq and every infinite path through T is a point in the Baire Space $\mathcal{N}$. For each $s \in Seq$, we take the tree:

$$T_s := \{t \in T \ : \ t \preceq s \text{ or } s \preceq t\}$$

That is, we trim all branches that don't pass through s. Using this new tree, we may define our new Cantor-Bendixson Derivative. Take:

$$T' := \{s \in T \ : \ f([T_s]) \text{ is uncountable } \}$$

Now we use ordinal recursion to define:

$$\begin{aligned} T^{(0)} &:= T \\ T^{(\alpha+1)} &:= (T^{(\alpha)})' \\ T^{(\beta)} &:= \bigcap_{\alpha<\beta} T^{(\alpha)} \qquad \text{if } \beta \text{ is a limit ordinal} \end{aligned}$$

As before, this process must stabilize eventually by a basic cardinality argument. We call the least ordinal α such that $T^{(\alpha)} = T^{(\alpha+1)}$ the "Cantor-Bendixson Rank" of A.

Now we must prove the same basic properties as before for our new Cantor-Bendixson construction. We start with a lemma:

Lemma 5.11: For every ordinal α, the tree $T^{(\alpha)}$ has no leaves. That is, for every $t \in T^{(\alpha)}$, there is an infinite path $p \in [T^{(\alpha)}]$ such that $t \prec p$.

Proof: We proceed by transfinite induction. We start by observing that $T^{(0)} = Seq$ trivially has no leaves.

Suppose that $T^{(\alpha)}$ has no leaves. Let $s \in T^{(\alpha+1)}$. By the construction, we know that $f([T_s^{(\alpha)}])$ is uncountable. Since Seq is countable, it follows there must be some $t \in T^{(\alpha)}$ such that $s \prec t$ and $f([T_t^{(\alpha)}])$ is uncountable. It follows that $t \in T^{(\alpha+1)}$ as well, so s is not a leaf. Therefore $T^{(\alpha+1)}$ has no leaves.

Finally suppose that β is a limit ordinal and $T^{(\alpha)}$ has no leaves for every $\alpha < \beta$. Then $T^{(\beta)} = \bigcap_{\alpha<\beta} T^{(\alpha)}$ trivially has no leaves. □

Proposition 5.12: The Cantor-Bendixson Rank of an analytic set is a countable ordinal.

Proof: Since the Baire Space $\mathcal{N}$ is Polish, we know that there is a countable basis $\{U_n\}_{n\in\mathbb{N}}$ for the topology on $\mathcal{N}$. Further by part 4 of proposition 5.8, we see that for every ordinal α, the infinite paths $[T^{(\alpha)}] \subseteq \mathcal{N}$ form a closed set. Since the sequence of $T^{(\alpha)}$ must stabilize eventually, it follows that $[T^{(\alpha)}]$ must stabilize eventually as well. By Lemma 5.11, we know that $T^{(\alpha)}$ is leaf free for all α. Hence for $\alpha < \gamma$, since $T^{(\gamma)}$ is a leaf-free sub-tree of the leaf-free tree $T^{(\alpha)}$, there must be some infinite path $p \in [T^{(\alpha)}]$ such that $p \notin [T^{(\gamma)}]$. Hence we see that $[T^{(\gamma)}] \subsetneq [T^{(\alpha)}]$. Since these sets are closed and $\mathcal{N}$ has a countable basis, it immediately follows that the construction must stabilize at a countable ordinal. Hence the Cantor-Bendixson Rank of A is countable. □

Proposition 5.13: For all ordinals α, let R_α (R stands for "removed") denote the set of points:

$$R_\alpha := \bigcup_{s \in T^{(\alpha)} \setminus T^{(\alpha+1)}} f([T_s^{(\alpha)}])$$

R_α is at most countable.

Proof: Notice that since $T^{(\alpha)} \setminus T^{(\alpha+1)}$ is a subset of *Seq*, we are taking a union of countably many sets. Further, for every $s \in T^{(\alpha)} \setminus T^{(\alpha+1)}$, we know that $f([T_s^{(\alpha)}])$ is a countable set by definition. Hence R_α is a countable union of countable sets, which is countable. □

Finally we may prove the capstone theorem of this section:

Theorem 5.9 (reprint): The analytic sets in Polish spaces satisfy the Perfect Set Property.

Proof: Let A be an uncountable analytic set in a Polish space X. Let $\text{rk}(A)$ denote the Cantor-Bendixson Rank of A, and $T^* := T^{(\text{rk}(A))}$. Let $R := A \setminus f([T^*])$. It is easy to see that:

$$R \subseteq \bigcup_{\alpha < \text{rk}(A)} R_\alpha$$

By Propositions 5.12 and 5.13, we know that $\text{rank}(A)$ is a countable ordinal and each R_α is a countable set. Thus R is countable. Since A is uncountable meanwhile, it follows that $f([T^*])$ is uncountable. We will show that $f([T^*])$ has a perfect subset.

Fix an arbitrary $s \in T^*$. I claim there are $t_1, t_2 \in T^*$ such that $s \prec t_1, t_2$, but $f([T^*_{t_1}])$ and $f([T^*_{t_2}])$ are disjoint. Since $f([T^*])$ is uncountable, we may pick infinite paths $p_1, p_2 \in [T^*]$ such that $f(p_1) \neq f(p_2)$. Since f is continuous, the topology on $\mathcal{N}$ allows us to find $t_1, t_2 \in T^*$ such that $t_1 \prec p_1$, $t_2 \prec p_2$, and $f(O(t_1))$ is disjoint from $f(O(t_2))$, where $O(t_j) = \{p \in \mathcal{N} : t_j \prec p\}$. Since $[T^*_{t_j}] = [T^* \cap O(t_j)]$, we get the desired result.

By applying the above argument inductively, we may construct a sequence of points in T^* indexed by finite binary sequences $\{s_b : b \in 2^{<\omega}\}$ with the following properties:

1. If $b_1 \preceq b_2$, then $s_{b_1} \prec s_{b_2}$.
2. If b_1 and b_2 are finite binary sequences such that $b_1 \not\preceq b_2$ and $b_2 \not\preceq b_1$, then $f([T^*_{b_1}])$ and $f([T^*_{b_2}])$ are disjoint.

From this set, we may construct a new sub-tree $U \subseteq T^*$ by:

$$U := \{s \in T^* : s \preceq s_b \text{ for some finite binary sequence } b\}$$

Notice right away that every node $s \in U$ has exactly one or two immediate successors. It follows that the obvious map $2^\omega \to [U]$ is a homeomorphism. Further, our map f such that $f(\mathcal{N}) = A$ is an injection when restricted to $[U]$. Hence applying our utility lemma 3.9, we see that $f([U]) \subseteq A$ is perfect. □

Corollary 5.14: Analytic sets in $\mathbb{R}$ satisfy the Continuum Hypothesis.

Remark 5.15: Interestingly, we came full circle through these arguments. Our arguments got more interesting and sophisticated as we moved away from Cantor and Bendixson's proof for closed subsets of $\mathbb{R}$ and into Polish spaces, G_δ sets, and Borel sets. When it came to analytic sets though, we ultimately had to use the same basic approach as Cantor and Bendixson!

6 The Bottom of the Rabbit Hole

We have now shown that analytic sets (and thus Borel sets) have the Perfect Set Property. It is worth pointing out now that the analytic sets do not form a $\mathbf{\Sigma}$-algebra. While the analytic sets are closed under countable unions and intersections, the compliment of an analytic set need not be analytic. This gives us a natural idea of a "co-analytic" set – a set whose complement is analytic. Continuing this line of thinking naturally leads to the notion of the "Projective Hierarchy".

Definition 6.1: Let X be a Polish space. Then we define the "Projective Hierarchy" on X as follows:

- $\mathbf{\Sigma}^1_1$ is the class of analytic subsets of X.
- $\mathbf{\Pi}^1_1$ is the class of sets X with analytic complement is analytic. These are known as the "co-analytic sets".
- For $n \in \mathbb{N}$, a subset $A \subseteq X$ is $\mathbf{\Sigma}^1_{n+1}$ if there is a $\mathbf{\Pi}^1_n$ subset B of the Polish space $X \times X$ such that A is the first coordinate projection of B.
- For $n \in \mathbb{N}$, a subset $A \subseteq X$ is $\mathbf{\Pi}^1_{n+1}$ if there is a $\mathbf{\Sigma}^1_n$ subset B of the Polish space $X \times X$ such that A is the first coordinate projection of B.
- For $n \in \mathbb{N}$, a subset $A \subseteq X$ is $\mathbf{\Delta}^1_n$ if it is both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.

Remark 6.2: It turns out that $\mathbf{\Delta}^1_1$ is exactly the class of Borel sets in X. This major result is due to Suslin.

Looking at $\mathbb{R}$ as a Polish space, how far up the Projective Hierarchy can the Perfect Set Program go? That is, for which n do $\mathbf{\Pi}^1_n$ and $\mathbf{\Sigma}^1_n$ sets have the Perfect Set Property?

In the early 1900s, the French analysts were working on this problem to no avail. No one was able to make any headway on even showing that $\mathbf{\Pi}^1_1$ sets have the Perfect Set Property. After years of failed attempts, Luzin allegedly remarked that "One does not know and one will never know" whether co-analytic sets have the Perfect Set Property.

Luzin turned out to be exactly right. Like the Continuum Hypothesis itself, the statement "co-analytic sets in $\mathbb{R}$ have the Perfect Set Propertry" is independent of ZFC. In 1938, Gödel published the following theorem:

Theorem 6.3 (Gödel, 1938): In ZFC + V=L, there is an uncountable $\mathbf{\Pi}^1_1$ set of real numbers that does not contain a perfect subset.

That is, there is a specific model of ZFC in which the co-analytic sets do not have the perfect set property.

Meanwhile, following Cohen's work on Forcing, set theorists were able to build bizarre models of set theory in which all kinds of interesting properties hold. In 1970 Robert Solovay published the following theorem:

Theorem 6.4 (Solovay, 1970): Suppose there is a transitive ϵ-model of ZFC + "There is an inaccessible cardinal". Then there is a transitive ϵ-model of ZF in which every uncountable set of reals contains a perfect subset.

Solovay's model shows that there is a model of set theory in which the Perfect Set Property holds – not just for the co-analytic sets, but for all sets! The only caveat is that Solovay's model doesn't satisfy the axiom of choice. In fact though, we can find a model of ZFC in which the co-analytic sets have the Perfect Set Property. Indeed, for a set A, let $L[A]$ denote the sets constructable from A. $L[A]$ is a model of ZFC set theory. Suppose we have a model where for every $A \subseteq \omega$, we have that $\aleph_1^{L[A]}$ is countable. That is, the cardinal $\aleph_1$ relativized to $L[A]$ is countable. One may show that such a model exists by Levy Collapse, and that such a model satisfies that every uncountable $\mathbf{\Sigma}^1_2$ (and hence $\mathbf{\Pi}^1_1$) subset of the reals has a non-empty perfect subset. For more details on this construction, look at theorem 25.38 in Jech. We summarize with the following theorem:

Theorem 6.5: There is a model of ZFC in which the co-analytic sets have the Perfect Set Property.

Thus the Perfect Set Program bottoms out at $\mathbf{\Sigma}^1_1$. We did pretty well though! While there are only $2^{\aleph_0}$ analytic subsets of $\mathbb{R}$, this class is still extremely general. In fact, I bet I can name the only two non-analytic sets you've ever worked with right now...

7 Two Pathological Sets

In all likelihood, you have seen exactly two examples of non-analytic sets. These are Vitali sets and Hamel bases for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Let's review these quickly.

Vitali sets: Take $\mathbb{R}$ as an additive group, and consider the quotient group $\mathbb{R}/\mathbb{Q}$. The standard Vitali set V is formed by taking exactly one representative of every coset. Some use "Vitali set" to more generally refer to any set with one representative of each equivalence class of $\mathbb{R}/D$, where D is a dense countable subgroup of the additive group of reals. We will be using the standard Vitali set, but nearly everything we say will generalize. You likely encountered the Vitali set as an example of a non-measurable set in a real analysis course.

Hamel bases for $\mathbb{R}$ as a $\mathbb{Q}$-vector space: Consider the real numbers as a vector space over $\mathbb{Q}$. A Hamel basis H for this vector space is a collection of real numbers such that for every $r \in \mathbb{R}$, there is a unique finite collection $h_1, \cdots, h_n \in H$ and $q_1, \cdots, q_n \in \mathbb{Q}\setminus\{0\}$ such that $r = q_1h_1 + \cdots + q_nh_n$. You likely encountered Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ as a counter-intuitive consequence of the Axiom of Choice, or when studying fields in algebra.

Remark 7.1: Before anything else, let's observe that if V is a Vitali set and H is a Hamel basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space, then $|V| = |H| = |\mathbb{R}|$. To see this for V is easy. The defining property of Vitali sets let's us write $\mathbb{R}$ as a countable union of disjoint sets by:

$$\mathbb{R} = \bigcup_{v \in V} v + \mathbb{Q}$$

It follows that $|\mathbb{R}| = |V| \times |\mathbb{Q}|$, which implies that $|V| = |\mathbb{R}|$.

For H, we appeal to the fact that if $\mathbb{F}$ is an infinite field and V is a vector space over $\mathbb{F}$, then $|V| = \max\{|\mathbb{F}|, \dim(V)\}$. It follows that $|H| = |\mathbb{R}|$, since $\dim(\mathbb{R}) = |H|$.

These elementary cardinality arguments demonstrate that these sets cannot plausibly violate the Continuum Hypothesis in ZFC.

Vitali Sets and Hamel Bases are not Analytic

Let's discuss why Vitali sets and Hamel bases of $\mathbb{R}$ over $\mathbb{Q}$ are not analytic. These arguments rely on the following 1917 theorem of Luzin.

Theorem 7.2 (Luzin, 1917): Every analytic set in $\mathbb{R}$ is Lebesgue measurable.

We will not prove this theorem. The argument relies on the fact that every analytic set may be written as the continuous image of the Baire space $\mathcal{N}$, and that every set in $\mathbb{R}$ may be well approximated by a measurable set. For more detials see Jech theorem 11.18. The fact that Vitali sets are not analytic is an obvious corollary to the theorem.

Corollary 7.3: Every Vitali set in $\mathbb{R}$ is not analytic.

Next, let's look to Hamel bases for $\mathbb{R}$ over $\mathbb{Q}$. Perhaps surprisingly, we cannot use the measurabilty of analytic sets to prove that such a basis isn't analytic directly. There are bases that are Lebesgue measurable!

Proposition 7.4: There is a Hamel basis H for $\mathbb{R}$ as a $\mathbb{Q}$ vector space such that H is Lebesgue measurable.

Proof: Take $\mathcal{C}$ to be the standard middle-thirds Cantor set. Recall that $\mathcal{C}$ consists of exactly those real numbers in the interval $[0,1]$ with a ternary expansion consisting of all 0s and 2s. It immediately follows that $\mathcal{C}+\mathcal{C}=[0,2]$. Hence the span of $\mathcal{C}$ is $\mathbb{R}$ when viewing $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Thus there is a Hamel basis $H \subseteq \mathcal{C}$. This Hamel basis will have outer measure 0 by the monotonicity of outer measure, making H Lebesgue measurable. □

Remark 7.5: While a Hamel basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space can be measurable, there is a nice way to build a non-measurable set out of it. Let H be such a Hamel basis. Index the elements of H by $H=\{h_r : r \in [0,1]\}$. For each $x \in \mathbb{R}$, let $a_0(x)$ denote the coefficient on h_0 in the H-expansion of x. For every $r \in [0,1]$, set:

$$A_0 := \{x \in \mathbb{R} : a_0(x) = 0\}$$

What can we say about A_0? It will be illuminating to look at the case where $h_0 = 1$. In this case, we see that $\text{span}(h_0) = \text{span}(1) = \mathbb{Q}$. So for every $x \in \mathbb{R}$, there is a rational number $q \in \mathbb{Q}$ and $y \in A_0$ such that $x = q + y$. Further, since A_0 is closed under addition, we see for all $x, y \in A_0$, $x - y \notin \mathbb{Q}$. These two properties together tell us that A_0 contains exactly one member of each coset of $\mathbb{Q}$ as a subgroup of $\mathbb{R}$. That is, A_0 is a Vitali set!

In the event that $h_0 \neq 1$, the exact same argument holds except $\text{span}(h_0) = h_0\mathbb{Q}$ and we get a generalized Vitali set instead. In any event, A_0 will be non-measurable for the exact same reasons as a Vitali set is.

With this method for constructing a non-measurable set out of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$, we may proceed with our argument.

Proposition 7.6: Let H be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. H is not analytic.

Proof: The following proof is reproduced from Nadkarni and Sunder (https://www.imsc.res.in/ sunder/mgnvss.pdf). Denote the elements of H by $H = \{h_r : r \in [0,1]\}$. For sake of contradiction, suppose that H is analytic. Let $A := H \setminus \{h_0\}$. For $n \in \mathbb{N}$ and $\vec{q} \in \mathbb{Q}^n$, define the map $f_{\vec{q}} : A^n \to \mathbb{R}$ by:

$$f_{\vec{q}}(a_1, \cdots, a_n) = \sum_{j=1}^{n} q_j a_j$$

Since H is analytic, we see that A is analytic, and that A^n is analytic as a subset of $\mathbb{R}^n$. Hence $f_{\vec{q}}(A^n) \subseteq \mathbb{R}$ is analytic, since f is continuous.

Now for each $x \in \mathbb{R}$, let $a_r(x)$ denote the coefficient on h_r in the H expansion of x. Let $A_0 := \{x \in \mathbb{R} : a_0(x) = 0\}$. Recall from remark 7.5 that A_0 is a generalized Vitali set, and hence non-measurable. Notice that $x \in A_0$ if and only if there is an $n \in \mathbb{N}$ and $\vec{q} \in \mathbb{Q}^n$ such that $x \in f_{\vec{q}(A^n)}$. Hence we may write:

$$A_0 = \bigcup_{n \in \mathbb{N}} \bigcup_{q \in \mathbb{Q}^n} f_{\vec{q}(A^n)}$$

So A_0 is a countable union of analytic sets, and thus is analytic itself. But then Luzin's theorem guarantees that A_0 is measurable. This is a contradiction. Hence H must not be analytic. □

Vitali Sets and Hamel Bases with Perfect Subsets

At the beginning of this essay I made the claim that you have probably never seen an uncountable set of reals which provably does not contain a perfect subset. Both Vitali sets and Hamel bases for $\mathbb{R}$ as a $\mathbb{Q}$-vector space seem like they may challenge my assertion. They are not analytic, so we cannot apply our above work. Let's make sure I wasn't lying. We prove a pair of propositions.

Proposition 7.7: There is a Vitali set $V \subseteq \mathbb{R}$ that contains a perfect subset.

To prove this, we will need a pair of lemmas:

Micro Lemma 7.8: Let I_1 and I_2 be disjoint compact intervals in $\mathbb{R}$. Let $q \in \mathbb{Q}$. Then there are compact intervals $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$ such that for all $x, y \in J_1 \cup J_2$, the difference $x - y \neq q$.

Proof: Let $\ell(\cdot)$ denote the length of an interval. Without loss of generality, suppose that I_1 is to the left of I_2. First, take J_1 to be a compact sub-interval of I_1 such that $\ell(J_1) < \min\{q, \ell(I_2)\}$. Since $\ell(J_1) < q$, for all $x, y \in J_1$ we have that $x - y \neq q$.

Now we translate J_1 by q to get $J_1 + q = \{x + q : x \in J_1\}$. Since $\ell(J_1) < \ell(I_2)$, we know that $(J_1 + q) \not\supset I_2$. It follows that $I_2 \setminus (J_1 + q)$ is either all of I_2, or a half-open sub-interval of I_2. Either way, we may take a compact sub-interval $J_2 \subseteq I_2 \setminus (J_1 + q)$ such that $\ell(J_2) < q$. Then by construction, for $x, y \in J_2$ and $z \in J_1$, we have that $x - z \neq q$ and $x - y \neq q$. This proves the lemma. $\square$

Lemma 7.9: Let $I_1, \cdots, I_n$ be disjoint compact intervals in $\mathbb{R}$. Let $q \in \mathbb{Q}$. Then there are disjoint compact intervals $J_1, \cdots J_{2n}$ such that $J_{2i-1}, J_{2i} \subseteq I_i$ for all i, and for all $x, y \in J_1 \cup \cdots \cup J_{2n}$, the difference $x - y \neq q$.

Proof: This is a situation where the construction is best expressed in pseudo-code. Treat the micro lemma 7.8 (ML) as a subroutine. For two disjoint compact intervals I_1 and I_2, let a call to the subroutine $\mathrm{ML}(I_1, I_2)$ output disjoint compact intervals (J_1, J_2) such that $J_1 \subseteq I_1$, $J_2 \subseteq I_2$, and for all $x, y \in J_1 \cup J_2$ the difference $x - y \neq q$.

Let $I_1, \cdots, I_n$ be disjoint compact intervals in $\mathbb{R}$. we replace these intervals with new, smaller subintervals $I_1, \cdots I_n$ by running the following program:

```
i := 1
  while i < n
       j := i + 1
       while j ≤ n
          (I_i, I_j) := ML(I_i, I_j)
           j := j + 1
       end while
       i := i + 1
  end while
return (I_1, ..., I_n)
```

After running this program, notice that our new I_j is a compact sub-interval of our original I_j, but now for all $x, y \in I_1 \cup \cdots \cup I_n$, the difference $x - y \neq q$.

To finish off the proof of the lemma, for each j take any two disjoint compact subintervals $J_{2j-1}, J_{2j} \subseteq I_j$. The collection $J_1, \cdots, J_{2n}$ are the desired intervals. $\square$

Proof of Proposition 7.7: Say that a set $X \subseteq \mathbb{R}$ has the "irrational difference property" if for all $x, y \in X$, the difference $x - y \notin \mathbb{Q}$. Notice that for any such X, there is a Vitali set V containing X. Since uncountable closed sets have the Perfect Set Property, it will suffice to find any uncountable closed set C with the irrational difference property. We may expand such a C to a Vitali set.

Let $I_1^0 := [0, 1]$, and let $\{q_1, q_2, \cdots\} = \mathbb{Q}$ be an enumeration of the rationals. Now using our lemma, for all $n \in \mathbb{N}$, we may construct disjoint compact intervals $I_1^n, \cdots, I_{2^n}^n$ such that:

1. I_{2j-1}^{n+1} and I_{2j}^{n+1} are sub-intervals of I_j^n.
2. For all $x, y \in I_1^{n+1} \cup \cdots \cup I_{2^{n+1}}^{n+1}$, the difference $x - y \neq q_{n+1}$.

Set $C_n := I_1^n \cup \cdots \cup I_{2^n}^n$, and set $C := \bigcap_n C_n$. This Cantor-set-like construction is clearly uncountable and closed. Now suppose that C does not have the irrational difference property. Then there are $x, y \in C$ such that $x - y = q_n$ for some n. But C is a subset of C_n which violates our construction. Hence C has the rational difference property. This completes the proof. $\square$

Proposition 7.10: There is a Hamel Basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space that contains a perfect subset.

Proof: The following construction comes from a paper of F. B. Jones. Take an enumeration of the rationals by $\mathbb{Q} = \{r_1, r_2, \cdots\}$. Without loss of generality, take $r_1 = 0$. Construct a series of closed intervals $\{I_{n,m} : n \geq 1,\ 1 \leq m \leq n\}$ with the following properties:

1. $I_{1,1} \not\ni 0$.
2. For each n, m, the intervals $I_{n+1,2m-1}$ and $I_{n+1,2m}$ are disjoint subintervals of $I_{n,m}$ with $I_{n+1,2m-1}$ lying entirely to the left of $I_{n+1,2m}$.
3. For each n, given $x_1, \cdots, x_n \in \bigcup_{m=1}^{2^{n-1}} I_{n,m}$, the following holds:

$$q_1x_1 + \cdots + q_nx_n \text{ with each } q_i \in \{r_1, \cdots, r_n\} \implies q_1 = \cdots = q_n = r_1 = 0$$

This construction can be formalized in a similar way to how we did for the Vitali set. Now take:

$$M := \bigcap_n \left[\bigcup_{m=1}^{2^{n-1}} I_{n,m} \right]$$

It is clear that M is a perfect set. By our construction, it is also clear that M is a set of $\mathbb{Q}$-linearly independent points in $\mathbb{R}$. M may be extended to a Hamel basis H for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. $\square$

So prior to this post, you have seen two sets where are not analytic. However, there are still Vitali sets and Hamel bases of $\mathbb{R}$ over $\mathbb{Q}$ that contain perfect subsets. It follows that such sets on their own are not going to be useful for showing the futility of the Perfect Set Program. We'll need something even more pathological...

8 The Most Pathological Sets I Know: Bernstein Sets

Let's introduce a third kind of pathological set:

Definition 8.1: A subset $B \subseteq \mathbb{R}$ is a "Bernstein set" if and only if for every uncountable closed $C \subseteq \mathbb{R}$, we have that $C \cap B$ is non-empty, but $C \not\subseteq B$. That is, B touches every uncountable closed set, but contains none of them.

Notice that t is not immediately clear that Bernstein sets even exist! I encourage you to think about why such a set exists before you read further. We present the existence of a Bernstein set as a theorem:

Theorem 8.2: There is a Bernstein set in $\mathbb{R}$.

Proof: Let $\mathcal{C} := \{C \subseteq \mathbb{R} : C \text{ is closed and } |C| = |\mathbb{R}|\}$. It is easy to see that $|\mathcal{C}| = 2^{\aleph_0}$. Take $\mathcal{C} = \{C_\alpha\}_{\alpha<\gamma}$ to be an indexing by the ordinals, with γ being the smallest ordinal of size $2^{\aleph_0}$.

By transfinite induction, define:

1. $A_0 := \emptyset$ and $B_0 := \emptyset$
2. $A_{\alpha+1} := A_\alpha \cup \{a_\alpha\}$ and $B_{\alpha+1} := B_\alpha \cup \{b_\alpha\}$ for every $\alpha < \gamma$, where a_α and b_α are distinct points chosen from $C_\alpha \setminus (A_\alpha \cup B_\alpha)$.
3. $A_\beta := \bigcup_{\alpha<\beta} A_\alpha$ and $B_\beta := \bigcup_{\alpha<\beta} B_\alpha$ if β is a limit ordinal.

Why is this well defined? For all $\alpha < \gamma$, we have that $|A_\alpha| = |B_\alpha| = |\alpha|$ since we add in exactly one point at every step in the construction. Since γ is the first ordinal with the same cardinality as $\mathbb{R}$, we also have that $|\alpha| < |\mathbb{R}|$ for all $\alpha < \gamma$. So $|C_\alpha| = |\mathbb{R}|$ and $|A_\alpha \cup B_\alpha| < |\mathbb{R}|$, we have that $|C_\alpha \setminus (A_\alpha \cup B_\alpha)|$ is infinite. So we can always find a measly two points!

Set $B := \cup_{\alpha<\gamma} B_\alpha$. By construction, it is clear that for each α we have:

$$b_\alpha \in B \cap C_\alpha$$

so B intersects every element of $\mathcal{C}$. On the other hand, since $a_\alpha \notin B$ by our construction, we have that $C_\alpha \not\subseteq B$. Thus B is a Bernstein set. $\square$

Remark 8.3: This exact proof works in an arbitrary Polish Space.

Bernstein sets are a bit weird. It will do us good to briefly run through a bunch of properties of these sets. Note that some of these properties (like uncountability) are obvious for the Bernstein set we constructed in the proof of Theorem 8.3, but are not obvious directly from the definition of a Bernstein set.

Proposition 8.4: Let $B \subseteq \mathbb{R}$ be a Bernstein set. B has the following properties:

1. $\mathbb{R} \setminus B$ is a Bernstein set as well.
2. The Lebesgue outer measure of a Bernstein set is infinite.
3. B is not Lebesgue measurable.
4. B is uncountable.
5. B contains no non-empty perfect subset.
6. B is not analytic nor co-analytic.
7. B does not have the "Property of Baire".

Proof:

1. By definition, for any uncountable closed C, there must be two points $c_1, c_2 \in C$ such that $c_1 \in B$ and $c_2 \in \mathbb{R} \setminus B$. Thus $\mathbb{R} \setminus B$ touches every uncountable closed set but contains none of them as well.
2. Let $m_*(\cdot)$ denote the Lebesgue outer measure. Suppose that $m_*(B) < \infty$. We may find an open set $U \supseteq B$ such that $m_*(U) < \infty$. Set $C := \mathbb{R} \setminus U$. Notice that since U is open, C is closed. U and C are both measurable, and $\mathbb{R} = U \cup C$, so $m_*(C) = \infty$. It follows that C is uncountable. But this makes C an uncountable closed subset of $\mathbb{R} \setminus B$. This contradicts property 1, as $\mathbb{R} \setminus B$ is a Bernstein set.
3. Suppose C is a closed subset of B. Then C must be countable by definition. It follows that the Lebesgue inner measure of B is zero. But by part 2, we have that the outer measure is infinite. Thus B must be non-measurable.
4. This follows immediately from property 2.
5. Every non-empty perfect set in $\mathbb{R}$ is uncountable and closed, and thus not a subset of B.
6. Since analytic sets have the Perfect Set Property, part 5 tells us that B cannot be analytic. Since $\mathbb{R} \setminus B$ is also a Bernstein set by part 1, part 5 says it also cannot be analytic. Thus B cannot be co-analytic either.
7. For a set A to have the Property of Baire means there is an open set U such that the symmetric difference $A \Delta U$ is meager. For sake of contradiction, Suppose that B has the Property of Baire. One may show that sets with the Property of Baire form a σ-algebra, so $\mathbb{R} \setminus B$ also has the property. The Baire cateogry theorem ensures $\mathbb{R}$ is not meager as a subset of itself. Since the union of meager sets is meager, it follows that either B or $\mathbb{R} \setminus B$ are not meager. Suppose that B is not meager. Then we may write $B = U \cup M$ where U is non-empty and open, and M is meager. The Baire category theorem ensures there is an uncountable G_δ subset of $\mathbb{R}$ contained in the difference $U \setminus M$. By our past work though, this means that G contains a non-empty perfect subset $P \subseteq \mathbb{R}$. So $P \subseteq B$, which is a contradiction to the definition of a Bernstein set since perfect sets are uncountable and closed. The same argument works if $\mathbb{R} \setminus B$ is not meager, since $\mathbb{R} \setminus B$ is a Bernstein set by property 1. □

Remark 8.5: These properties hold in any uncountable Polish space.

Remark 8.6: Bernstein sets thus demonstrate the futility of the Perfect Set Program. Every Bernstein set is uncountable, but demonstrably contains no perfect subset.

So what makes Bernstein sets so pathological? Early descriptive set theorists were largely concerned with three "regularity properties" of subsets of $\mathbb{R}$. These regularity properties were the Perfect Set Property, Lebesgue measurability, and the Property of Baire. Proposition 8.4 shows that Bernstein sets lack all three of these properties! They are, in a sense, *fully* pathological.

Vitali Sets and Hamel Bases Redux

In section 7 of this essay we saw that despite not being analytic, there are Vitali sets and Hamel bases for $\mathbb{R}$ as a $\mathbb{Q}$ vector space that contain non-empty perfect subsets. Is it true that all such sets contain perfect subsets? Now that we have seen Bernstein sets, we have the tools to answer this question.

Proposition 8.7: There is a Bernstein set which is also a Vitali set.

Proof: Recall that a set $X \subseteq \mathbb{R}$ is said to have the "irrational difference property" if and only if all $x, y \in X$, the difference $x - y \notin \mathbb{Q}$. For any set $C \subseteq \mathbb{R}$ with $|C| < 2^{\aleph_0}$, the set of rational translations $C + \mathbb{Q} := \{c + q : c \in C, q \in \mathbb{Q}\}$ also has cardinality less that $2^{\aleph_0}$. Copying our construction of a Bernstein set in Theorem 8.2, the prior observation allows us to construct a pair of Bernstein sets A and B such that $A \cap B = \emptyset$, and both A and B have the irrational difference property. We simply must select our points a_α and b_α so as to avoid creating a rational difference.

Since A has the irrational difference property, A contains at most one element of every coset of $\mathbb{R}/\mathbb{Q}$. We may thus extend A to a Vitali set $V \supseteq A$ by tossing in an arbitrary representative of all the missing cosets. We claim that V is a Bernstein set.

To see that V is a Bernstein set, we simply must prove V contains no uncountable closed subset. Take any non-zero rational q, and consider the set $V + q$. Since V is a Vitali set, we have $V \cap (V + q) = \emptyset$. Hence we find that $V \subseteq \mathbb{R} \setminus (V + q) = (\mathbb{R} \setminus V) + q$. Since $\mathbb{R} \setminus V$ is contained in $\mathbb{R} \setminus A$, which is Bernstein by Proposition 8.4, $\mathbb{R} \setminus V$ contains no uncountable closed subset. Translating by a rational will not change this, so we see that V is a Bernstein set. □

Proposition 8.8: There is a Bernstein set which contains a Hamel basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space.

Proof: In this article by Kysiak (https://www.degruyter.com/document/doi/10.2478/s11533-009-0053-0/html), a method for cooking up Bernstein sets with desired algebraic properties is presented. As an example of the power of the method, Kysiak cooks up a Bernsetein set B such that $B + B = B$ and $B - B = \mathbb{R}$. Since $B - B \subseteq \text{span}(B)$, we see there must be a basis H contained inside B. □

Previously we showed that Vitali sets and Hamel bases for $\mathbb{R}$ as a $\mathbb{Q}$-vector space can contain perfect subsets. Thus, it would be fruitless to try and prove that an *arbitrary* Vitali set or Hamel basis could break the Perfect Set Program. Meanwhile, It follows from Propositions 8.7 and 8.8 that specific constructions of these pathological sets may be counterexamples! However, showing this required us to, more or less, construct a Bernstein set that happened to have extra properties. Is there a way to construct a Vitali set or a Hamel basis that doesn't have a non-empty perfect subset without using this technology? If so, we could demonstrate the futility of the Perfect Set Program without relying on Bernstein sets! At the time of writing though, I am unaware of such a construction.

Bernstein Sets and the Continuum Hypothesis

At this point, we know that Bernstein sets are uncountable and have no non-empty perfect subsets. This takes a powerful tool for proving a set has the cardinality of the continuum off the table. Could it be that there are Bernstein sets with intermediate cardinalities? No it could not.

Proposition 8.7: Let $B \subseteq \mathbb{R}$ be a Bernstein set. B has the cardinality of the continuum.

Proof: It will clearly suffice to find a collection $\mathcal{A}$ of disjoint uncountable closed subsets of $\mathbb{R}$ with $|\mathcal{A}| = |\mathbb{R}|$. To this end, let $S := \{0, 2\}^\omega$ denote the collection of all countably infinite sequences of

0s and 2s. Let $f : S \to \mathbb{R}$ be the "ternary encoding" operation:

$$f(s_1, s_2, s_3, \cdots) = [0.s_1 s_2 s_3 \cdots]_3 = \sum_{j=1}^{\infty} \frac{s_j}{3^j}$$

where $(s_1, s_2, s_3, \cdots) \in S$. By identifying $\{0, 2\}^\omega$ with the Cantor Space 2^ω in the obvious way, we may view f as a homeomorphism onto the Cantor set.

Now let $I : S \times S \to S$ denote the "interweaving operation":

$$I(s, t) = (s_1, t_1, s_2, t_2, s_3, t_3, \cdots)$$

where $s = (s_1, s_2, s_3, \cdots)$ and $t = (t_1, t_2, t_3, \cdots)$ are sequences in S.

Fix a sequence $t \in S$. Let $I_t : S \to S$ by $I_t(s) := I(s, t)$. It is easy to see that I_t is a homeomorphism onto its image. So for every t, we have that $I_t(S) \cong 2^\omega$. Hence $f \circ I_t(S)$ is a homeomorphic copy of the Cantor set in $\mathbb{R}$. Finally we notice that for $t \neq r$, that $f \circ I_t(S)$ and $f \circ I_r(S)$ are disjoint.

It follows from this reasoning that $\mathcal{A} := \{f \circ I_t(S) \ : \ t \in S\}$ is a collection of continuum-many disjoint uncountable closed subsets of $\mathbb{R}$. Since our Bernstein set B must contain at least one point from each $A \in \mathcal{A}$, it follows that $|B| = |\mathbb{R}|$. $\square$

This conclusion should give us pause. Even Bernstein sets provably have the cardinality of the continuum! They might not have the Perfect Set Property, but they still aren't even possibly violations of the Continuum Hypothesis. We've proven the futility of the Perfect Set Program, but nothing more! We would need to work even harder to come up with a description of a set that even plausibly violates the continuum hypothesis.

9 Consequences and Conclusions

Let's recap our work. In section 1, we introduced perfect sets and proved that non-empty perfect sets in $\mathbb{R}$ always have the cardinality of the continuum. Consequently, perfect sets presented a path toward proving the Continuum Hypothesis, dubbed the Perfect Set Program. Over the next few sections, we fleshed out a variety of concepts and tools, most notably Cantor-Bendixson analysis, in order to show that uncountable sets of increasing complexity necessarily contain non-empty perfect subsets. This culminated with a proof that every analytic set satisfies the Continuum Hypothesis in section 5. In section 6, we saw how work of Gödel and Solovay showed that in the absence of additional axioms, we cannot say whether or not the co-analytic sets have the Perfect Set Property. In sections 7 and 8, we looked at Vitali sets, Hamel bases, and Bernstein sets. All three of these types of sets are extremely pathological and are necessarily non-analytic. Vitali sets and Hamel bases may or may not contain non-empty perfect subsets, while Bernstein by definition cannot contain a non-empty perfect subset. Thus of these three classes of sets, only Bernstein sets truly show the futility of the Perfect Set Program. However, none of these sets could even possibly have a cardinality strictly between those of the naturals and the reals.

I feel comfortable saying that we have proven our thesis that virtually every uncountable set we mathematicians come across in the wild has the cardinality of the continuum. We had to work really extremely hard to find sets without the Perfect Set Property, and even then these sets still had the cardinality of the continuum. How should we respond to this finding?

Empiricist Set Theory?

In some sense, the success of the Perfect Set Program can be "empirical evidence" for the Continuum Hypothesis. When we come across an infinite subset of $\mathbb{R}$ in a wild, we should take this as evidence that "most likely" it is either countable or has the full cardinality of the continuum. Every example of a topological space we work with will not be of an intermediate cardinality. Further, suppose we have some proposition that we can prove is true if we assume the Continuum Hypothesis. Then this empirical evidence suggests that we should still be able to apply the result, even if we don't want to assume the Continuum Hypothesis... Ok kind of. You should probably still check everything. But it's interesting! I'm curious if there is a way to make this idea formal. Is there a way to put a probability distribution on the power set of the reals that puts a more probability mass on classes of sets of lower complexity?

But we don't need to rely on notions of likelihood. We're set theorists, and it is our God given right to pick our own damn axioms. If we engage in some inductive reasoning, perhaps the success of the Perfect Set Program should be taken as evidence that in the "true" set theory–God's set theory–the Continuum Hypothesis is true. From the human perspective, our choice of the axioms of set theory are up to our tastes, of course, but this certainly pushes my tastes closer to "the continuum hypothesis is true". It's worth noting that even when $V = L$, the situation in which the Perfect Set Program breaks down at $\mathbf{\Pi}^2_1$, the Continuum Hypothesis holds! While not formal, I think the Perfect Set Program provides an interesting rationale for accepting the Continuum Hypothesis as an additional "standard axiom" of set theory.

10 Applications

There's one last loose end to tie up. Way back in the introduction, I said that the pursuit of the Perfect Set Program provided mathematicians with new and powerful tools. If you made it this far in this essay, I think you will agree that these tools are interesting. But useful? Really?

Yes, really. The techniques of Cantor-Bendixson analysis can be used to great effect in Model Theory. All of these examples come from David Marker's book "Model thoery: An Introduction". Basic familiarity with model theory is assumed.

Application 1: Cantor Bendixson Analysis on Stone Spaces

Definition 9.1: Let T be a theory over a language L. An "n-type" in T is a collection $\tau(x_1, \cdots, x_n)$ of formulas in L with free variables among $x_1, \cdots, x_n$ such that $\mathrm{Th}(A) \cup \tau(x_1, \cdots, x_n)$ is satisfiable with a consistent assignment of the $x_1, \cdots, x_n$ free variables. An n-type $\tau(x_1, \cdots, x_n)$ is "complete" if and only if for every formula $\phi(x_1, \cdots, x_n)$, either ϕ or $\neg\phi$ is in τ. We let $S_n(T)$ denote the set of all complete n-types over T.

Remark 9.2: These "types" are different than the kinds of types we deal with in type theory. These types are about categorizing different "kinds" of elements in a structure by looking at all the different formulae they satisfy. Indeed, for any element a in a structure M, we may generate a complete type in $S_1(\mathrm{Th}(M))$ by looking at $\{\phi(x) : M \vDash \phi(a)\}$.

Example 9.3: Consider the complete first order theory of arithmetic $T := \mathrm{Th}(\langle \mathbb{N}, 0, S, +, \cdot, < \rangle)$. We may define $\mathbb{P} := \{n \in \mathbb{N} : n \text{ is prime }\}$ inside of T. Enumerate the primes sequentially as $\mathbb{P} = \{p_1, p_2, \cdots\}$. For any $p \in P$, let $p \mid x$ be a shorthand for the formula:

$$p \mid x := \text{"}(\exists y)(y \cdot p = x)\text{"}$$

That is, $p \mid x$ asserts that x is divisible by p. Let $F \subseteq \mathbb{P}$ be a finite set of primes, and consider the following collection of formulae:

$$\tau(x) := \{p \mid x \ : \ p \in F\} \cup \{p \nmid x \ : \ p \notin F\}$$

Is τ a type? Yes! To check this, we need to see that there is some model of T in which there is an element that simultaneously satisfies all of the formulae in τ. But this is easy! Take $\langle \mathbb{N}, 0, S, +, \cdot, < \rangle \vDash T$, and assign the product of primes $\prod_{p \in F} p$ to the variable x.

What if we instead took $F' \subseteq \mathbb{P}$ to be an infinite collection of primes, and the analogous $\tau'(x)$? We can no longer multiply infinitely many primes together to show that τ' is a type. However, every finite subset of the sentences in $\tau'(x)$ is satisfiable, so we may apply the compactness theorem to assert that there is a non-standard model of arithmetic that satisfies τ'. Hence τ' is still a type. Since there must further be a countable model realizing this type, the $2^{\aleph_0}$ different choices of F illustrate that there are at least $2^{\aleph_0}$ non-isomorphic countable models of the complete first order theory of arithmetic.

Construction 9.4: For a theory T, we may endow the complete n-types of T with a topology. Indeed, for any formula ϕ in n-free variables, let $[\phi]$ denote the collection:

$$[\phi] := \{\tau \in S_n(T) \,:\, \phi \in \tau\}$$

That is, $[\phi]$ is the collection of all the different complete n-types that contain ϕ as a formula. Take the collection of such sets as a basis for a topology on $S_n(T)$. One may easily show the following lemma:

Lemma 9.5: Let T be a theory, and endow $S_n(T)$ with the topology from construction 9.4. The topology has the following properties:

1. For all formulae ϕ and ψ in the same n free variables, we have that $[\phi \wedge \psi] = [\phi] \cap [\psi]$, $[\phi \vee \psi] = [\phi] \cup [\psi]$, and $[\neg\phi] = S_n(T) \setminus [\phi]$.
2. Every $[\phi]$ is clopen.
3. $S_n(T)$ is compact, Hausdorff, and totally disconnected.

A topological space that is compact, Hausdorff, and totally disconnected is called a "Stone space". It follows from the lemma that every space of complete types is a Stone space. Notice that Stone spaces are not necessarily Polish, so our prior Cantor Bendixson work does not apply. We can extend our analysis though:

Construction 9.5 (Cantor-Bendixson Rank for Stone Spaces): Let T be a Stone space. Let T' denote the collection of non-isolated points in T. We now define by transfinite induction:

$$\begin{aligned} T^{(0)} &:= T \\ T^{(\alpha+1)} &:= \left(T^{(\alpha)}\right)' \text{ for all ordinals } \alpha \\ T^{\beta} &:= \bigcap_{\alpha<\beta} T^{(\alpha)} \text{ for limit ordinals } \beta \end{aligned}$$

One may verify that each $T^{(\alpha)}$ is a Stone space. Let T^{∞} denote the collection of points that are in $T^{(\alpha)}$ for all ordinals α. Further, we easily see that there must be some minimal α such that $T^{(\alpha)} = T^{\infty}$. We call this ordinal the "Cantor-Bendixson rank" of T, and denote it by $\text{rk}(T)$. Further, we may show that T^{∞} is a perfect set. We use these ideas to prove the following theorem:

Theorem 9.6 (Cantor-Bendixson Theorem for Stone Spaces): Let T be a Stone space, and let $\text{Clopen}(T)$ denote the collection of clopen sets in T. If $|T| \gneq |\text{Clopen}(T)|$, then $|T| \geq 2^{\aleph_0}$.

Proof Sketch: First, we note that $|T \setminus T^{\infty}| \leq |\text{Clopen}(T)|$. To see this, suppose that $x \in T$ is removed at stage $\alpha+1$. Then there is a clopen set C such that x is the unique element in $C \cap T^{(\alpha)}$. This gives us an injection from $T \setminus T^{\infty}$ into $\text{Clopen}(T)$.

It follows that since $|T| \gneq |\text{Clopen}(T)|$, we have that $T^{\infty} \neq \emptyset$. It is also easy to duplicate our prior proofs and show that a perfect Stone space has cardinality at least as large as the continuum. Thus $|T| \geq |T^{\infty}| \geq 2^{\aleph_0}$. □

This theorem has a handful of striking corollaries:

Corollary 9.7: Suppose that T is a countable theory. Then $|S_n(T)| \leq \aleph_0$ or $|S_n(T)| = 2^{\aleph_0}$.

Corollary 9.8: If T has an ω-saturated model, then T has an atomic model.

Corollary 9.9: Suppose that T is a "small" theory, meaning that for all n, $|S_n(T)| \leq \aleph_0$. Then for all n, viewing $S_n(T)$ as a Stone space, the isolated points are dense in $S_n(T)$.

Application 2: Morley Rank

Let M be an L-structure. For any $A \subseteq M$, let L_A be the language where we have extended L by a constant symbol for every $a \in A$. Interpret M as an L_A structure in the obvious way. An n-types on M in with parameters from A is a collection of L_A-formulae $\tau(x_1, \cdots, x_n)$ with the same free variables such that $Th(M) \cup \tau$ is satisfiable. We denote the space of complete n-types over M with parameters from A by $S_n^M(A)$. This notion allows us to extend our analysis of types from theories to models.

Definition 9.10: Let M be a structure. M is "ω-saturated" if and only if for all finite $A \subseteq M$ and $\tau(\overline{x}) \in S_n^M(A)$, the type τ is realized in M. That is, there is a tuple $\overline{m} \in M^n$ such that $\tau = \{\phi : M \vDash \phi(m)\}$.

Definition 9.11: Let T be a complete theory. T is "ω-stable" if and only if for all models $M \vDash T$ and all $A \subseteq M$ with $|A| = \omega$ we have $|S_n^M(A)| = \omega$.

In a sense, saturation is telling us that our model is big enough to realize all its types, and stability tells us that that our theory is simple enough that we don't have "too many types". The concept of "Morley rank" is a powerful tool for analyzing such models and theories that gives us a sense of "dimension".

Definition 9.12: Let M be an L structure and $\phi(v_1, \cdots, v_n)$ be an L_M formula. The "Morely rank" of ϕ in M, denoted $\mathrm{RM}^M(\phi)$, is defined transfinitely as follows:

1. $\mathrm{RM}^M(\phi) = -1$ if and only if $\phi[M] = \{\overline{m} \in M^n : M \vDash \phi(\overline{m})\} = \emptyset$.
2. $\mathrm{RM}^M(\phi) \geq 0$ if and only if $\phi[M] \neq \emptyset$. That is, ϕ is satisfiable in M.
3. $\mathrm{RM}^M(\phi) \geq \alpha + 1$ for ordinal α if and only if there are L_M formulae $\psi_1(\overline{v}), \psi_2(\overline{v}), \cdots$ such that the collection of $\psi_j[M]$ are pairwise disjoint subset of $\phi[M]$, and each $\mathrm{RM}^M(\psi_j) \geq \alpha$.
4. $\mathrm{RM}^M(\phi) \geq \alpha$ for limit ordinal α if and only if $\mathrm{RM}^M(\phi) \geq \beta$ for all $\beta < \alpha$.
5. $\mathrm{RM}^M(\phi) = \infty$ if and only if $\mathrm{RM}^M(\phi) \geq \alpha$ for all ordinals α.

Remark 9.13: Notice that this definition is extremely cumbersome and applies to all formulae in all structures. We may clean this up considerably by restricting our attention to models of ω-saturated models. We present the following proposition:

Proposition 9.14: Let A be a structure, ϕ be a formula with parameters from A, and let M and N be ω-saturated elementary extensions of A. The following hold:

1. $\mathrm{RM}^M(\phi) = \mathrm{RM}^N(\phi)$.
2. If M' is another ω-saturated model such that $M \preceq M'$, then $\mathrm{RM}^M(\phi) = \mathrm{RM}^{M'}(\phi)$.

This proposition allows us to simplify our definition of Morley rank in the following way. Since saturated extension always exist, we may simply work in a highly saturated model and remove the model dependence. For details, see section 6.2 of Marker on the Monster Model.

Morley rank has applications in Algebraic Geometry. Here's an example:

Theorem 9.15: Let K be an algebraically closed field, and $V \subseteq K^n$ an irreducible algebraic geometry. Then $\mathrm{RM}(V)$ is the Krull dimension of V.

How does this connect to Cantor Bendixson analysis? Well, let T be an ω-stable theory and let $M \vDash T$. Considering $S_n^M(M)$ as a closed subset of itself as a Stone space, we may compute the Cantor-Bendixson rank of any $p \in S_n^M(M)$. That is, the unique ordinal α such that p is in $(S_n^M(M))^{(\alpha)} \setminus (S_n^M(M))^{(\alpha+1)}$ (or ∞ if there is no such α). We present the following theorem:

Theorem 9.16: Let T be an ω-stable theory and let $M \models T$. For any complete $\tau \in S_n^M(M)$, let rank(τ) denote the Cantor-Bendixson rank of τ as an element of $S_n^M(M)$. Then:

$$\text{rank}(\tau) = \min\{\text{RM}(\phi) : \phi \in \tau\}$$

Thus Cantor-Bendisxon analysis gives us a useful computational tool in Algebraic Geometry!

Application 3: The Number of Countable Models

Here's a fun question to work through. Suppose we have a complete theory T over a countable language. Up to isomorphism, how many distinct countable models can T have? Call this quantity $N(T)$. By the Lowenheim-Skölem theorem, we know that $N(T) \geq 1$. Looking at any countably categorical theory (like the theory of dense linear orders without endpoints), we see that $N(T)$ can be exactly 1. Playing around with constants and unary predicates, it's also easy to see that for every finite $n \geq 3$, there are theories such that $N(T) = n$. It is a surprising result of Vaught that there is no theory T such that $N(T) = 2$.

What about the infinite cases? Looking at the theories of an algebraically closed field of characteristic $p < \infty$, and the first order theory of the field of real numbers, we find theories for which $N(T) = \aleph_0$ and $N(T) = 2^{\aleph_0}$. In the absence of the Continuum Hypothesis, can we have a theory T with $\aleph_0 < N(T) < 2^{\aleph_0}$? Surprisingly, this is an open problem! According to Marker, the best known result is a result of Morley:

Theorem 9.17 (Morley): If $N(T) > \aleph_1$, then $N(T) = 2^{\aleph_0}$.

The details of this proof require going into infinitary logic, a very interesting topic that deserves a much more full discussion than I can provide here. Sweeping these details under the rug, the proof boils down to showing that for all theories T, a cetain class of types can always be identified with an analytic subset of the Cantor space 2^ω. Thus the Perfect Set Property for analytic sets gives us that this class is either countable, or has cardinality of the continuum. This technique can't eliminate the possibility of having $\aleph_1$ models, but it eliminates all other potential intermediate cardinalities.

11 Appendix 1: The Definition of "Analytic Sets" and the Borel Isomorphism Theorem

We have two competing definitions of Analytic in play. We claim they are equivalent. Consider the following theorem:

Theorem A1.1: Let X be a Polish space, and $A \subseteq X$. The following are equivalent:

1. There is a Polish space Y and a Borel set $B \subseteq X \times Y$ such that $A = \pi B$, where π is the first coordinate projection.
2. There a Borel set $B \subseteq X \times X$ such that $A = \pi B$, where π is the first coordinate projection.

Recall that item 1 was taken in section 5 to be our definition for A to be an analytic set in X. Meanwhile, item 2 is the "implied" definition of analytic from looking at analytic sets in $\mathbb{R}$. We need new technology to prove this theorem.

Definition A1.2: Let X and Y be Polish spaces, and let $f : X \to Y$. The function f is "Borel measurable" if and only if for every Borel set $B \subseteq Y$, $f^{-1}(B)$ is Borel in X. f is "Borel bi-measurable" if and only if f is Borel measurable and for every Borel set $B \subseteq X$, the image $f(B)$ is Borel in Y. Finally f is a "Borel isomorphism" if and only if f is a Borel bi-measurable bijection. In this case we say such X and Y are "Borel isomorphic", and write $X \cong_B Y$.

The following theorem lives at the center of the theory of stochastic processes.

Theorem A1.3 (Borel Isomorphism Theorem): Let X and Y be Polish spaces. $X \cong_B Y$ if and only if X and Y have the same cardinality.

Our technique for proving this theorem fleshes out the ideas in this paper of Rao and Srivastava . We start with some lemmas.

Lemma A1.4: Let X and Y be Polish spaces, and $f : X \to Y$ be a be a map such that $f^{-1}(U) \subseteq X$ is Borel for every open $U \subseteq Y$. Then f is Borel measurable.

Proof: Consider the following collection of sets in Y:

$$\mathcal{A} := \{A \subseteq Y \;:\; f^{-1}(A) \text{ is Borel in } X\}$$

By hypothesis, every open subset of Y is in $\mathcal{A}$. Suppose $A \in \mathcal{A}$. We know $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$. It follows that $Y \setminus A$ is in $\mathcal{A}$, so $\mathcal{A}$ is closed under complements. Further suppose that $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of sets in $\mathcal{A}$. Then $f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n)$. It follows that $\bigcup_n A_n \in \mathcal{A}$, so $\mathcal{A}$ is closed under countable unions. Ergo $\mathcal{A}$ is a σ-algebra containing the open sets, so every Borel set of Y is in $\mathcal{A}$. This proves that for every Borel set $B \subseteq Y$, we have that $f^{-1}(B)$ is Borel in X. Hence f is Borel measurable. □

Lemma A1.5: Let X be a Polish space. There is a Borel bi-measurable injection from X into the Hilbert cube $[0,1]^\omega$.

Proof: If X is finite or countable, every $x \in X$ is closed. Any subset $S \subseteq X$ can be written as a an at-most-countable union of closed set, meaning every subset of X is Borel. Since every finite or closed subset of $[0,1]^\omega$ is Borel, we may simple take any injection $f : X \to [0,1]^\omega$.

If X is uncountable, let $S \subseteq X$ be a countable dense subset of X, and let d metrize the topology on X. Without loss of generality, suppose $d < 1$. Take $f : X \to [0,1]^\omega$ by:

$$f(x) := \big(d(x,s_1),\ d(x,s_2),\ d(x,s_3),\ \cdots\big)$$

where $s_1, s_2, s_3, \cdots$ is an enumeration of S. It is easy to see that f is an injection and a homeomorphism onto its image. It follows from lemma A1.4 that f is a bi-measurable injection. □

Lemma A1.6: The Hilbert cube I^ω is Borel isomorphic to the Cantor Space 2^ω.

Proof: Let D denote the set of dyadic rationals in $[0,1]$. That is, those rational numbers whose denominators are powers of two when written in lowest terms. Include 0 and 1 in D. Let $f : [0,1] \setminus D \to 2^\omega$ by taking $f(x)$ to be the coefficients of the binary expansion of x. Notice that the image of f is exactly those sequences in 2^ω that are not eventually constant. It is clear that f is a homeomorphism onto its image.

Let $F : [0,1] \to 2^\omega$ be any bijection extending f. First, I claim that F is Borel measurable. Let $U \subseteq 2^\omega$ be open. Then we may decompose:

$$F^{-1}(U) = F^{-1}(U \cap F([0,1] \setminus D)) \cup F^{-1}(U \cap F(D))$$

Since $F = f$ on $[0,1] \setminus D$ and f is a homeomorphims onto its image, it is clear that $F^{-1}(U \cap F([0,1] \setminus D))$ is open in $[0,1] \setminus D$. On the other hand, since D is countable and F is a bijection, we have that $F^{-1}(U \cap F(D))$ is at most countable, and thus is Borel. It easily follows that $F^{-1}(U)$ is Borel. By lemma A1.4, F is Borel measurable. A virtually identical argument to the above shows that for any open set $U \subseteq [0,1]$, the image $F(U)$ is Borel in 2^ω. Hence since F is invertible, lemma gives us that F is Borel bi-measurable.

It follows that there is a Borel bi-measurable bijection $F^\omega : [0,1]^\omega \to (2^\omega)^\omega$. We see by Currying that $(2^\omega)^\omega$ is homeomorphic to $2^{\omega\times\omega}$, which in turn is homeomorphic to 2^ω by the standard bijection of $\omega \times \omega$ to ω. But it is clear by lemma 5.7 that a homeomorphism is Borel bi-measurable. By composing, we get a Borel isomorphism from I^ω to 2^ω. □

Lemma A1.7 (Bi-Measurable Cantor-Schroder-Bernstein): Let X and Y be Polish spaces. Suppose there are Borel bi-measurable injections $f : X \to Y$ and $g : Y \to X$. Then X and Y are Borel isomorphic.

Proof: For any $S \subseteq X$, take a set $H(S) := X \setminus g(Y \setminus f(X))$. Now take $A_0 = \emptyset$, and for each $n \in \mathbb{N}$, take $A_{n+1} := H(A_n)$. Now set $A := \bigcup_n A_n$. Notice easily that each A_n is Borel in X, so A is Borel. Second, notice by elementary set arithmetic that:

$$H(A) = X \setminus g(Y \setminus f(\cup_n A_n)) = \bigcup_n X \setminus g(Y \setminus f(\cup_n A_n)) = \bigcup_n A_{n+1} = A$$

It follows that $g(Y \setminus f(A)) = X \setminus A$. So we may define a bijection $h : X \to Y$ by:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in X \setminus A \end{cases}$$

Since f and g^{-1} are Borel bi-measurable bijections on the Borel sets A and $X \setminus A$ respectively, it follows that h is a Borel isomorphism. □

These utility lemmas are all we need to prove the Borel isomorphism theorem.

Proof of Theorem A1.3 (Borel Isomorphism Theorem): Let X and Y be Polish spaces. The forward direction for the Borel isomorphism theorem is trivial, so we only need to show the reverse direction. Suppose $|X| = |Y|$. We have two cases to consider:

Case 1: X is finite or countable. For every $x \in X$, $\{x\}$ is closed. Any subset $S \subseteq X$ can therefore be written as an at-most-countable union of closed sets, meaning every subset of X is Borel. Similarly, every subset of Y is Borel. Take any bijeciton $f : X \to Y$. It follows easily that f is a Borel isomorphism.

Case 2: X is uncountable. By lemmas A1.5 and A1.6, we may take a Borel bi-measurable injection $f : X \to 2^\omega$. On the other hand, since X is an uncountable Polish space, Utility lemma 3.9 gives us a homeomorphic embedding $g : 2^\omega \to X$. By lemma A1.4, it follows that g is a Borel bi-measurable injection. By bi-measurable Cantor-Schroder-Bernstein (lemma A1.6), we get that X and 2^ω are Borel isomorphic.

Repeating the exact same argument, we get that $Y \cong_B 2^\omega$ as well. Composing maps gives us $X \cong_B Y$.

This completes the proof. □

Remark A1.8: As mentioned before, this theorem lies at the heart of Stochastic processes. Most of the time we are interested in real valued random variables. So let X be such a random variable. Formally, X is a measurable function from some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ denotes the Borel σ-algebra on $\mathbb{R}$. In general, it doesn't hurt to use Borel σ-fields (and refinements of them) on Polish spaces as the underlying probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$. But we almost always treat the underlying space as $\mathbb{R}$ (For example, when constructing a standard Brownian motion). The Borel isomorphism theorem explains why we can do this without losing any generality!

Remark A1.9: One may further show that if X and Y are Polish spaces, and $f : X \to Y$ is a Borel measurable bijection, then f is a Borel isomorphism. This result can be found in Kechris's book on descriptive set theory.

We may now return to our goal of reconciling the two definitions of analytic. Recall we defined an analytic set in $\mathbb{R}$ as the projection of a Borel set in $\mathbb{R} \times \mathbb{R}$ (definition 5.1), but an analytic set in a Polish space X as the projection of a Borel set in $X \times Y$ where Y is another Polish space. We stated but had not proved a proposition that resolves this conflict:

Theorem A1.1 (Reprint): Let X be a Polish space, and $A \subseteq X$. The following are equivalent:

1. There is a Polish space Y and a Borel set $B \subseteq X \times Y$ such that $A = \pi B$, where π is the first coordinate projection.

2. There a Borel set $B \subseteq X \times X$ such that $A = \pi B$, where π is the first coordinate projection.

Proof: It is immediately clear that 2 implies 1, so we only need to prove the other direction. Suppose that A is analytic (in the sense of 1). We have two cases. If X is finite or countable, then $X \times X$ is a countable Polish space and every subset is Borel. Hence we may take $A \times X \subseteq X \times X$ to be our Borel set and observe that $\pi(A \times X) = A$.

Now consider the case where X is uncountable. Since A is analytic, there is a Polish space Y and a Borel set $B \subseteq X \times Y$ such that $A = \pi B$. By lemma 5.8 and the Borel isomorphism theorem, we may find a Borel bi-measurable injection $f : Y \to Z$. Now take a function $F : X \times Y \to X \times Z$ by $F(x, y) = (x, f(y))$. It is easy to check directly that F is Borel bi-measurable. Hence $F(B) \subseteq X \times Z$ is Borel. It is also clear that $\pi F(B) = A \subseteq X$. $\square$

12 Appendix 2: The Role of Choice

How does the Axiom of Choice impact this work? What is and isn't dependent on choice? Without going into too much detail, let's discuss some big ideas.

Section 1 (Perfect Sets): non-empty perfect sets in Polish spaces always have cardinality of the continuum, even without choice. While it's a little annoying, all the "choice functions" needed can be stated explicitly.

Sections 2-5 (Perfect Set Property for Closed, G_δ, Borel, and Analytic Sets): The core results of these sections may be recovered with some effort. For example, Hartogs's theorem may be used to recover Lemma 2.5. In particular, we recover that Analytic sets have the Perfect Set Property in the absence of choice.

Section 6 (Co-Analytic Sets May Not Have the Perfect Set Property): This section demonstrates an interesting interplay between choice and the Perfect Set Program. Indeed, in when $V = L$, we see that there is an uncountable co-analytic set with no non-empty perfect subset. However, $V = L$ also implies the axiom of choice. Meanwhile, choice fails in Solovay's model, where all uncountable sets of reals have non-empty perfect subsets. Choice seemingly allows us to construct sets messy enough to lack the Perfect Set Property.

Sections 7 and 8 (Vitali Sets, Hamel Bases, and Bernstein Sets): These sections further demonstrate how choice gets in the way of the Perfect Set Program. In order to show that any of these three pathological classes of sets exist, we need some form of choice. Choice up to cardinality $2^{\aleph_0}$ is clearly sufficient.