Proofs as Rhetoric

In my experience, "rhetoric" — the art of persuasion — is a word that rarely comes up in conversations about mathematics. Perhaps the word conjures horrible memories of mandatory undergraduate writing courses and the essay on the GRE general test. This is a shame. As much as we may pretend that our proofs are rigorous, outside of proof assistants like Coq and Lean, proof-writing is, at its core, an exercise in effective persuasion. Mysteriously, this fact seems largely underappreciated by most mathematicians.

"Rhetoric" should be a part of our mathematical vocabulary. It's not a dirty word. Looking at proof-writing through the lens of rhetoric is a useful tool in all aspects of mathematics. As an instructor, focusing on the rhetorical aspects of proof-writing has improved my pedagogy and helped my students become more effective mathematics communicators themselves. As a researcher, it has allowed me to bridge gaps between different audiences that speak different mathematical dialects. Beyond concrete improvement of communication skills, this "Proofs as Rhetoric" perspective gives insight into the social nature of mathematics.

The Nature of Proof

Let's start by tearing off the Band-Aid: the proofs we write aren't rigorous. For clarity, when I speak of "proofs", I am talking about the kinds of proofs we express through natural language, read in books, write in papers, and present in talks. I am not talking about formal deductions à la sequent calculus or machine checkable proofs in Lean. It is the natural language proofs we write that are not rigorous. To make my case, I point your attention to the following delightful little Wikipedia page: "List of Incomplete Proofs." As this list shows, it is not uncommon for a published proof to later be deemed "incomplete", or even found to be a "proof" of a false result!

For interest, consider an example from Sanford L. Segal's excellent book, *Mathematicians Under the Nazis* [Seg03]. In 1931, Udo Wegner published a proof of a falsehood in *Mathematische Annalen*. This erroneous proof was discovered by Helmut Hasse, who published an explicit counterexample found by Emil Artin, together with a corrected-and-generalized version of Wegner's "theorem." A year later, Wegner republished a different correction of his original theorem with an additional hypothesis in the same journal. However, Bartel Leendert van der Waerden found and published another simple counterexample in 1933. Wegner had failed again.

How was it that both of Wegner's "proofs" made it through the review process and were published in a prestigious journal? These proofs may have failed as deductions, but they succeeded as arguments. They convinced the reviewers that the result was true. Herein lies the nature of proofs: proofs are not deductions, but arguments written to persuade the reader that the stated result is true. Proofs are rhetoric.

While the theory of rhetoric has advanced greatly over the last two-and-a-half millennia, it's rarely a poor choice to start with Aristotle. To the Greeks, there were three primary modes of persuasion: logos, ethos, and pathos. Let's step through these one by one, and see how they connect to mathematics.

Logos: an argument from reason. Unsurprisingly, logos holds a central role in mathematical argumentation. While our proofs are not formal, they are simulacra of formal arguments. Steps are skipped, arguments are left unstated, and mistakes are made, but generally statements in proofs logically follow from one another. Beneath the informal proof is the blueprint of a formal one.

Ethos: an argument from authority and credibility. The credentials and history of a speaker absolutely impact the way a proof is received. Consider the case of inter-universal Teichmüller theory (IUT). In 2012, acclaimed mathematican Shinichi Mochizuki released a series of long, impenetrable preprints that claimed to solve a number of outstanding conjectures in number theory and arithmetic geometry. Due to the length, writing style, and notation, mathematicians could not determine whether or not there were errors in these papers. In 2018, Fields Medalist Peter Scholze and Jakob Stix published a report claiming to have found a gap in the papers. Would Scholze and Stix have even bothered to respond to IUT if it were pioneered by a mathematician less accomplished than Mochizuki? Why do we treat some purported proofs of famous unsolved problems as being worth engaging with, and dismiss others as the musings of cranks? At the end of the day, these determinations come down to matters of credibility and authority. We initially screen arguments on the basis of the plausibility that the mathematician proved the result. This isn't a bad thing! We have limited time, and going through a proof takes effort. But recognize that this use of ethos is far from formal argumentation. This ethos also extends to citations. Why do I trust an appeal to a "private correspondence with Grothendieck", but not to a "private correspondence with my really smart friend"?

Pathos: an argument from emotion. While logos and ethos are present in mathematical arguments, surely there aren't appeals to emotion, right? Well, while I would certainly argue that overt emotional appeals are less prevalent than logical appeals and appeals to authority, I don't think mathematical writing is devoid of pathos. We may look to the proof of the four color theorem for an example. The four color theorem was originally proved with the help of a computer by an exhaustive case analysis of 1,834 different configurations. Despite being further reduced to a mere 633 cases and formalized inside of Coq, some mathematicians to this day consider the four color theorem an open problem. I personally know accomplished mathematicians with this view. I believe that viewing the proof as illegitimate is largely an emotional response. Mathematicians find something emotionally dissatisfying about the computer proof. Sure, it's a "proof", but it doesn't explain *why* the result is true in some intuitionistic sense. Does such an argument that lacks that emotionally powerful "Aha!" moment count as a proof? Perhaps if the machine proof of the four color theorem had itself been accompanied by a pathos-laden appeal to the value of machine-based proofs, it would have been widely accepted more quickly.

While this basic thesis that rhetoric impacts math is obvious, there are non-trivial corollaries. The rest of this essay will be dedicated to fleshing out some of these down-stream implications in research mathematics and teaching.

Rhetoric in Research Mathematics

If mathematical proofs are rhetorical in nature — not formal arguments but persuasive essays written to convince the reader of a result — then it necessarily follows that research mathematics is a social activity. The collection of "proven mathematical truths" is determined intersubjectively, existing only as a pseudo-consensus of qualified practitioners. So when writing a proof, it matters who exactly the proof is trying to persuade. Just like with a persuasive essay or speech, we tailor our proofs to the audience. We modulate the language we use, the concepts we employ, and the steps we skip depending on who we are writing the proof for. For example, I would explain my work with homotopy type theory (HoTT), to a logician differently than how I would explain it to a homotopy theorist. As an interdisciplinary field, HoTT is deeply connected to both homotopy theory and logic, but homotopy theorists and logicians approach the field from very different angles. Knowing your audience is arguably the most important part of rhetoric; different audiences have different social norms that must be respected and exploited to effectively convince them a result is true.

How do social norms impact the logos, ethos, and pathos of mathematical proof? Logos is perhaps the most interesting. Different audiences have different needs and desires for what it means for a proposition to “logically follow” from another. Here are two examples:

1. A more mathematically experienced audience can be trusted to fill in more gaps. “Experience” can be both local (knowledge in a narrow subject area) and global (the so-called mathematical maturity gained by having many touch-points across subjects). An audience with more local knowledge can be trusted to fill in more details on their own. This is similar to how a skilled chess player can think many moves ahead. Global experience, on the other hand, allows for effective argument by analogy. A mathematically mature audience can receive the structure of an argument, if not the details, merely through comparison to other results they know.

2. Communities can have radically different notions of rigor. Mathematicians pre-Cauchy debated if calculus proofs involving infinitesimals and the “ghosts of departed quantities” were rigorous. While some mathematicians wanted more structured proofs with well defined rules, others were happy to accept these arguments as rigorous. Today, this debate lives on in spirit when mathematicians poke fun at physicists for not checking convergence assumptions, representing different views of rigor between the math and physics communities. At the extreme of this dynamic, imagine there were a mathematical conjecture about the natural numbers with relevance for engineering. While the conjecture remains unproven, it has been checked up to 10^{100}. Numbers this large will never be encountered in the real world. To the engineers, the result is proven!

Some intersections of mathematical ethos and social norms are obvious. What credentials do we use to assess the authority of a mathematician? Why are there so few “outsider” autodidactic mathematicians, compared to philosophers and literary scholars? What institutions must a mathematician be a part of? Clearly today, the answer is university credentials and the university setting. This hasn’t always been the case. In the mid twentieth century, Bell Labs was a hub for the development of many mathematical advances. For example, Claude Shannon’s treatise *A Mathematical Theory of Communication* [Sha48], the founding of information theory, was written at Bell Labs. We can also look at appeals to so called “folklore results”. These are known facts within the community that no one has bothered to formally write up or publish in a cite-able way, but everyone in the tribe is aware of.

Other ways that social norms impact ethos are more subtle. Let’s return to the four color theorem. We may look at the continued refusal to accept the machine assisted proof through the lens of ethos. What authority does a computer have to prove a result true? How can we trust the computer when we can’t directly verify the integrity of the system? A rogue gamma ray could have flipped a bit at the very end! Should we treat such a proof as probabilistic? If so, should I also be treating all human proofs as probabilistic because there is some small-but-non-zero chance that I’m hallucinating right now?

Finally pathos. I believe that many mathematicians use pathos to achieve goals with their proofs other than proving a claim. For example, some mathematicians use overly complicated machinery and omit important details, making the paper harder to follow than it need be. I believe this is intentional. They are preying on our natural tendency to conflate “this paper is hard to understand” and “this result is hard to understand”. By doing so, the author makes their result seem more complicated, raising their prestige as a mathematician through the effective use of pathos. Indeed, Carl Linderholm makes this same point in his iconic book *Mathematics Made Difficult* when he says, “where there is no confusion there is no prestige. Mathematics is prestidigitation.” [Lin72]

Rhetoric in the Classroom

In the university setting, teaching is on the same order of importance as research. How does Proofs as Rhetoric apply in the lecture hall? In the standard undergraduate mathematics classroom, we see two kinds of proofs. There are the teacher-to-student proofs the professor gives in lecture and readings, and the student-to-teacher proofs the students submit in problem sets. The teacher-to-student proofs fit naturally into the Proofs as Rhetoric framework. Professors go to great lengths to tailor their arguments to the audience, and the proofs serve to convince the students the results are true.

At first glance though, the student-to-teacher proofs don't fit quite as nicely. When I assign a problem to my students, I already believe the result is true. Presumably I wouldn't have assigned it if I didn't. Further, the students are likely not at a level of mathematical maturity to appropriately tailor their arguments to me. Looking back at some of my own problem sets from undergrad, I would go into painful levels of detail about relatively mundane facts like $2^n > n$ for all $n \in \mathbb{N}$. The following observation hints at a resolution to this incongruity: when professors assign homework problems, they often select problems that give students other useful results that weren't covered in lecture. With this in mind, we see that the assigned proof isn't to convince the professor, but to convince the students. The students are writing for themselves! Perhaps more fully, the proof serves to the convince the professor that the students understand why the result is true.

This understanding has implications for grading in proof-based math courses. When I speak with graduate students in the humanities, they often express envy about how much easier it must be to grade mathematics problem sets than essays. Their reasoning centers around the claim that mathematics is *objective* while their essays are *subjective*. While I do not feel equipped to take a strong philosophical stance on the objectivity/subjectivity of mathematical truth itself, I will say with absolute certainty that the grading of mathematical proofs is not objective. When writing a proof, the student wants to convince the grader that they understand why a result is true. To that end, what convinces me as a grader may be different than what convinces you as a grader. After all, if math grading were objective, why would we have one TA grade the same problem on all the exams? How can students complain about tough graders and not just tough courses?

We have already established that when a student writes a proof on an exam or in a problem set, they are writing to convince themselves. Consequently, when an instructor grades these proofs, she tries to put herself in the position of a student, and pretend that she does not know what she knows. She dons a different veil of ignorance. In examining what makes an argument convincing from under the veil, I am drawn to a pair of countervailing effects. The first effect is "the things I know very well are obvious". As an illustration, this semester, I have been the teaching assistant for a real analysis class covering basic measure theory and Lebesgue integration. Through studying descriptive set theory, probability, and stochastic processes, I am very familiar with basic measure theory. There have been moments in grading where a student has made a true-but-unjustified statement about measure that is obvious to me and I haven't required them to explain. For example, that almost everywhere convergence implies convergence in measure. Upon reflection, I'm not sure the students actually understand why this is true! Perhaps I should have taken off points.

This effect is offset by a second consideration: "the things I know very well are hard and nuanced". Perhaps I'm just an egotist, but I'm drawn to the belief that the math I study is trickier than many give it credit for. Consequently I'm always looking for opportunities to point out the subtleties of the material I love. For example, in that same real analysis class, we had a problem set that focused on cardinality and the sizes of various sets. In one problem, the students were asked to prove that a certain set $X \subseteq \mathbb{R}$ could be put in one-to-one correspondence with the entire real line $\mathbb{R}$. The intended proof was to explicitly find a somewhat messy bijective. Several students instead turned in proofs that used the Baire category theorem to show that X was strictly larger than the natural numbers $\mathbb{N}$. While many would have given full marks, I didn't. The proof by Baire category theorem only works if we assume the Continuum Hypothesis is true! Much to the chagrin of David Hilbert, *uncountable* need not mean *cardinality of the continuum.* My interest in set theory primed me to notice this distinction, and I took the opportunity to emphasize a point of nuance.

In general, I think these two effects approximately cancel each other out. When I audit my own grading, the impact of "the things I know very well are obvious" and "the things I know very well are hard and nuanced" seem to be on the same order of magnitude. Which effect ultimately wins out on the margin likely varies from person to person though, so it's important to be aware of them both.

Proofs as Rhetoric offers another insight on grading beyond the intrinsic subjectivity. It justifies taking off *style points* for poorly written proofs. Many instructors want to take marks off for poorly written and confusing proofs, but feel they shouldn't because the argument is "mathematically correct". For example, a close colleague of mine is currently the teaching assistant for a complex analysis class. There was a homework problem involving computing an integral of a real-valued function on $\mathbb{R}$ using Cauchy's residue theorem. A student handed in a solution that was correct, but the final integral value was given as a messy expression containing the imaginary unit i. Ought this student receive full marks for the problem? In my opinion, no, they should not. This wasn't a mere violation of stylistic norms like leaving a fraction unreduced or a radical in the denominator. The final expression was very messy, and it was not clear at a glance that the imaginary parts would cancel out. After all, the integral of a real valued function must be real! How can the student claim to have convinced themselves they arrived at the correct answer when they don't even know if their answer is real?

This argument extends to other kinds of sloppy proofs. Consider the student who figuratively throws mud at the wall hoping that something sticks. This kind of proof is all too common. The student turns in a long list of statements as a proof. The list contains a proper sub-list that constitutes a correct and efficient proof. However, a large portion of the statements are irrelevant to the problem at hand. Should this proof receive full marks? Proofs as Rhetoric says "no!" By including all these extraneous statements, the proof fails to demonstrate that the student understands why the result is true. If they did understand, they would have recognized that these extra statements could be removed.

The Language of Mathematics

Mathematics is, at its core, a language. We use this language to understand formal deductions, to describe the geometry of spacetime, and to grope at infinity. But the map is not the territory, and the word is not the referent. While we may be able to use our intuitions and spatial reasoning to justify theorems to ourselves, it is not until we codify these intuitions in the language of mathematics that we can convince others. As researchers and instructors, we are bound by our limited fluency and rhetorical skill.

Consequently, mathematicians should work to improve their rhetoric. On some level, merely recognizing the rhetorical aspects of proof writing would be a big step in the right direction. A simple understanding of the ethos, logos, and pathos of mathematical argumentation will make for more convincing proofs in both journals and classroom lectures. We can do better, though. Mathematics is a language and we should make sure we teach that language fully. Different branches of math speak the same language, but it's as though they have different dialects. They describe the same concepts in different ways. If different adjective-ordering conventions[1] can give an indication of the attributes a language finds more important, different mathematical dialects can similarly indicate what a field finds important. Logicians, with their myriad shorthands for axiomatic systems, credibly signal the field's interest in minimal assumptions. Category theorists, on the other hand, despite studying many of the same systems as logicians, place greater emphasis on different notions of "equivalence", both mathematically and linguistically. Becoming rhetorically adept requires understanding these differences in dialect and the ability to seamlessly flow between them.

I don't claim that any of the ideas here are novel or groundbreaking[2]. Many of my arguments here can be reduced to the following quote from the mathematician/poet A. S. Yessenin-Volpin:

> I define proof as any fair way of making a sentence incontestable. Of course this explication is related to ethics – the notion of fair means 'free from any coercion and fraud' – and to the theory of disputes, indicating the cases in which a sentence is to be considered incontestable. [YV70]

Every working mathematician knows that the proofs we write aren't fully formal and that math is social. Every instructor knows the value of clear argumentative structure. I merely claim that looking at these ideas through the lens of rhetoric provides insight into the practice of mathematics. The Proofs as Rhetoric framework has been helpful to me as I write proofs for different audiences and teach different groups of people. Wider recognition of the rhetoric of proofs will make our work more intelligible to each other and outsiders. Finally, there is untapped value in telling our students about rhetoric up front. It's an opportunity to make math more clear for both researchers and students.

References

[Lak76] *Proofs and Refutations: The Logic of Mathematical Discovery.* Cambridge University Press, 1976.

[Lin72] Carl E. Linderholm. *Mathematics Made Difficult.* World Publishing, 1972.

[Seg03] Sanford L. Segal. *Mathematicians under the Nazis.* Princeton University Press, Princeton, N.J, 2003.

[Sha48] Claude Elwood Shannon. A mathematical theory of communication. *The Bell System Technical Journal*, 27:379–423, 1948.

[YV70] A.S. Yessenin-Volpin. The ultra-intuitionistic criticism and the antitraditional program for foundations of mathematics. In A. Kino, J. Myhill, and R.E. Vesley, editors, *Intuitionism and Proof Theory: Proceedings of the Summer Conference at Buffalo N.Y. 1968*, volume 60 of *Studies in Logic and the Foundations of Mathematics*, pages 3–45. Elsevier, 1970.

[1] Which sounds right in English: "big red balloon" or "red big balloon"?

[2] For an actual groundbreaking analysis of the social theory of informal proof, read Imre Lakatos's *Proofs and Refutations* [Lak76]. This is one of my favorite works of philosophy, and was the inspiration for this essay.